

DEFORMATIONS AND RIGIDITY OF LATTICES IN SOLVABLE LIE GROUPS

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ABSTRACT. Let G be a simply connected, solvable Lie group and Γ a lattice in G . The deformation space $\mathcal{D}(\Gamma, G)$ is the orbit space associated to the action of $\text{Aut}(G)$ on the space $\mathcal{X}(\Gamma, G)$ of all lattice embeddings of Γ into G . Our main result generalises the classical rigidity theorems of Mal'tsev and Saitô for lattices in nilpotent Lie groups and in solvable Lie groups of real type. We prove that the deformation space of every Zariski-dense lattice Γ in G is finite and Hausdorff, provided that the maximal nilpotent normal subgroup of G is connected. This implies that every lattice in a solvable Lie group virtually embeds as a Zariski-dense lattice with finite deformation space. We give examples of solvable Lie groups G which admit Zariski-dense lattices Γ such that $\mathcal{D}(\Gamma, G)$ is countably infinite, and also examples where the maximal nilpotent normal subgroup of G is connected and simultaneously G has lattices with uncountable deformation space.

1. INTRODUCTION

Let G be a simply connected real Lie group. A lattice Γ in G is *rigid* if every embedding of Γ into G as a lattice extends to an automorphism of the Lie group G . Landmark results about rigidity and superrigidity of lattices in the context of semisimple Lie groups are the Mostow Strong Rigidity Theorem and the Margulis Superrigidity Theorem.

In the context of solvable Lie groups, a classical theorem of Mal'tsev–Saitô states that, if G is nilpotent [14] or solvable of *real type* [23], i.e., if G is solvable and if the eigenvalues of all the transformations Ad_g , $g \in G$, in the adjoint action of G are real, then every lattice in G is rigid. On the other hand, it is known that generally lattices in solvable Lie groups can be very far from being rigid. In [26] Starkov gave important examples of rigid and non-rigid lattices in solvable Lie groups. In [31] Witte proved that, if G is solvable, then every Zariski-dense lattice Γ in G is superrigid in the following sense: any finite-dimensional representation $\varrho: \Gamma \rightarrow \text{GL}_n(\mathbb{R})$ virtually extends to a representation

Date: November 24, 2011.

2010 Mathematics Subject Classification. 22E40, 22E25, 20F16.

Key words and phrases. Solvable Lie groups, lattices, rigidity, deformations.

Acknowledgements. The authors thank the Deutsche Forschungsgemeinschaft and the London Mathematical Society for financial support.

of G . He also showed that, if G is solvable and Γ is a Zariski-dense lattice in G , then every homomorphic embedding of Γ into G as a lattice extends to a crossed automorphism of G , i.e., a certain type of ‘twisted’ automorphism.

In the present paper we initiate a quantitative description of the phenomenon of non-rigidity for lattices in Lie groups. Let

$$\mathcal{X}(\Gamma, G) = \{\varphi: \Gamma \hookrightarrow G \mid \varphi(\Gamma) \text{ is a lattice in } G\}$$

be the space of all homomorphic embeddings of Γ into G as a lattice, equipped with the topology of pointwise convergence. The space $\mathcal{X}(\Gamma, G)$ and its connected components feature in classical works of Weil [28] and Wang [27]. The group $\text{Aut}(G)$, consisting of all continuous automorphisms of G , is a Lie group and acts continuously on $\mathcal{X}(\Gamma, G)$ from the left via composition. We observe that Γ is rigid in G if and only if $\text{Aut}(G)$ acts transitively on $\mathcal{X}(\Gamma, G)$. More generally, the orbit space

$$\mathcal{D}(\Gamma, G) = \text{Aut}(G) \backslash \mathcal{X}(\Gamma, G),$$

provides a quantitative measure for the degree of non-rigidity of Γ in G . It can be interpreted as the *deformation space* of lattice embeddings of Γ into G . We note that the quotient space $\mathcal{D}(\Gamma, G)$ also reflects topological properties of the $\text{Aut}(G)$ -orbits in $\mathcal{X}(\Gamma, G)$.

Let G be a simply connected, solvable Lie group, and let Γ be a lattice in G . In general, deformation spaces of the form $\mathcal{D}(\Gamma, G)$ can be uncountable. In the present article we are particularly interested in describing principal situations where $\mathcal{D}(\Gamma, G)$ is finite or countable. The group G is said to be *unipotently connected* if its maximal nilpotent normal subgroup is connected. Our main theorem shows that Zariski-dense lattices in unipotently connected solvable Lie groups have finite deformation spaces.

The following discussion of our main results is built around three general themes: deformation spaces of lattices in unipotently connected groups, the characterisation of strong rigidity, and the topology of deformation spaces.

Deformation spaces for unipotently connected groups. We prove the following finiteness result for deformation spaces:

Theorem A. *Let G be a simply connected, solvable Lie group G which is unipotently connected. Then for every Zariski-dense lattice Γ of G the deformation space $\mathcal{D}(\Gamma, G)$ is finite. Moreover, its cardinality is uniformly bounded above by a constant depending only on the dimension of G .*

We remark that, if G is of real type, then G is unipotently connected and every lattice Γ in G is Zariski-dense. Hence, Theorem A generalises

the above mentioned rigidity theorems of Mal'tsev and Saitô. Furthermore, it yields the following application, showing that Γ is *weakly rigid* in G up to finite index in $\text{Aut}(\Gamma)$.

Corollary B. *Let G be a simply connected, solvable Lie group G which is unipotently connected, and let Γ be a Zariski-dense lattice of G . Then the automorphisms of Γ which extend to automorphisms of G form a finite index subgroup of $\text{Aut}(\Gamma)$. The index of this subgroup is uniformly bounded above by a constant depending only on the rank of Γ .*

The constant alluded to in Theorem A and Corollary B is specified in Section 8.1.

We emphasise that the conclusion of Theorem A ceases to hold if one of the assumptions, unipotent connectedness of the Lie group G or Zariski-denseness of the lattice Γ , is dropped. In Section 2 we construct a Zariski-dense lattice in a non-unipotently connected Lie group G which has a countably infinite deformation space. Moreover, we provide in the same section examples of non-Zariski-dense lattices in unipotently connected groups which have uncountable deformation spaces. A construction of Starkov shows that there exist rigid lattices which are not Zariski-dense; see [26, Example 6.1].

It is worth noting that unipotently connected Lie groups cover a wide range of lattices. Recall that every discrete subgroup of a solvable Lie group is polycyclic; see [18]. By a result of Auslander, every polycyclic group virtually embeds as a Zariski-dense lattice into a suitable simply connected, solvable Lie group; see [1, 31]. We show that, in addition, the Lie group can be taken to be unipotently connected; see Proposition 5.3. This refinement yields the following corollary.

Corollary C. *Let Γ be a polycyclic group. Then Γ admits a finite index subgroup Δ which embeds as a Zariski-dense lattice into a simply connected, solvable Lie group H such that the deformation space $\mathcal{D}(\Delta, H)$ is finite.*

The next corollary provides a simple structural criterion for a polycyclic group Γ to ensure that the deformation space associated to any Zariski-dense lattice embedding of Γ is finite.

Corollary D. *Let Γ be a polycyclic group such that the commutator subgroup $[\Gamma, \Gamma]$ has finite index in the Fitting subgroup $\text{Fitt}(\Gamma)$. Then for every Zariski-dense lattice embedding of Γ into a simply connected, solvable Lie group H the deformation space $\mathcal{D}(\Gamma, H)$ is finite.*

Indeed, the condition on Γ in Corollary D implies that every Zariski-dense lattice embedding of Γ is into a unipotently connected Lie group; see Proposition 6.6.

The condition of unipotent connectedness will be further motivated and explained in Section 5. The class of unipotently connected groups coincides with the class of groups (A) defined in [26]; the terminology we employ originates from [10]. If Γ is a Zariski-dense lattice in G , then G is unipotently connected if and only if the dimension of the nilradical of G is equal to the rank of the Fitting subgroup of Γ ; see Corollary 5.4. The Fitting subgroup $\text{Fitt}(\Gamma)$ is the maximal nilpotent normal subgroup of Γ . This illustrates that unipotently connected groups afford strong ‘structural’ links to their Zariski-dense lattices.

Strong rigidity and the structure set. Let G be a simply connected, solvable Lie group. A Zariski-dense lattice Γ in G is called *strongly rigid* if every embedding of Γ as a Zariski-dense lattice into a simply connected, solvable Lie group H extends to an isomorphism of Lie groups $G \rightarrow H$. Zariski-denseness guarantees that such extensions are unique, whenever they exist. While rigidity of Γ in G is a property which crucially depends on the ambient Lie group G , strong rigidity only depends on the group Γ itself. Indeed, we consider the *structure set* for Zariski-dense embeddings of Γ into simply connected, solvable Lie groups H , defined as

$$\mathcal{S}^Z(\Gamma) = \{\varphi: \Gamma \hookrightarrow H \mid \varphi(\Gamma) \text{ is a Zariski-dense lattice in } H\} / \sim,$$

where two embeddings $\varphi_1: \Gamma \hookrightarrow H_1$ and $\varphi_2: \Gamma \hookrightarrow H_2$ are equivalent if there exists an isomorphism of Lie groups $\psi: H_1 \rightarrow H_2$ such that $\psi \circ \varphi_1 = \varphi_2$. Clearly, Γ is strongly rigid if and only if $\mathcal{S}^Z(\Gamma)$ consists of a single element.

For every simply connected, solvable Lie group G such that Γ is a Zariski-dense lattice in G , the deformation space $\mathcal{D}(\Gamma, G)$ embeds naturally into the structure set $\mathcal{S}^Z(\Gamma)$; see Section 8.3. Our principal observation concerning the structure set $\mathcal{S}^Z(\Gamma)$ is the following.

Theorem E. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . Then the structure set $\mathcal{S}^Z(\Gamma)$ is either countably infinite or it consists of a single element. The structure set consists of a single element if and only if Γ embeds as a lattice into a simply connected, solvable Lie group of real type.*

Corollary F. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . Then $\mathcal{D}(\Gamma, G)$ is at most countably infinite.*

In particular, Theorem E shows that Γ is strongly rigid if and only if Γ is a lattice in a simply connected, solvable Lie group of real type. Combining Theorem A and Theorem E, we prove the following dichotomy result for Zariski-dense lattices in unipotently connected groups.

Corollary G. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G which is unipotently connected. Then either Γ*

is strongly rigid or there exist countably infinitely many pairwise non-isomorphic (unipotently connected) simply connected, solvable Lie groups which contain Γ as a Zariski-dense lattice.

Topology of the deformation space. Let G be a simply connected, solvable Lie group, and let Γ be a lattice in G . We apply our results to study the structure of the space $\mathcal{X}(\Gamma, G)$ of lattice embeddings of Γ into G and, in particular, the properties of the $\text{Aut}(G)$ -action on $\mathcal{X}(\Gamma, G)$. Recall that $\mathcal{X}(\Gamma, G)$ is a subspace of the space $\text{Hom}(\Gamma, G)$ of all homomorphisms of Γ into G , equipped with the topology of pointwise convergence. The group $\text{Aut}(G)$, which naturally carries the structure of a Lie group, acts continuously on $\mathcal{X}(\Gamma, G)$, and the deformation space

$$\mathcal{D}(\Gamma, G) = \text{Aut}(G) \backslash \mathcal{X}(\Gamma, G),$$

equipped with the quotient topology, reflects properties of the $\text{Aut}(G)$ -orbits in $\mathcal{X}(\Gamma, G)$.

Since Γ is finitely generated, the space $\text{Hom}(\Gamma, G)$ carries the additional structure of a real algebraic variety; in particular, it has only finitely many connected components. By a celebrated result of Weil [28], for any cocompact lattice Δ in a Lie group H the space $\mathcal{X}(\Delta, H)$ is an open and locally path connected subset of $\text{Hom}(\Delta, H)$. In classical situations, for example, in the case $H = \text{PSL}(2, \mathbb{R})$, the space $\mathcal{X}(\Delta, H)$ coincides with the union of two connected components of $\text{Hom}(\Delta, H)$; see [7, 8]. Moreover, the set $\mathcal{X}(\Delta, H)$ and its quotient, the Teichmüller space, are connected manifolds which can be described by algebraic equalities and inequalities; see [13].

For the simply connected, solvable Lie group G a result of Wang [27] implies that the connected components of $\mathcal{X}(\Gamma, G)$ are manifolds. However, for a general solvable Lie group H , even if Δ is a Zariski-dense lattice in H , the space $\mathcal{X}(\Delta, H)$ can have infinitely many connected components; see Example 2.5. This shows that, in general, $\mathcal{X}(\Delta, H)$ cannot be described as a semi-algebraic subset of $\text{Hom}(\Delta, H)$.

In contrast to the general picture, Theorem A shows that for a large class of lattices the space $\mathcal{X}(\Gamma, G)$ has only finitely many connected components. Indeed, whenever $\mathcal{D}(\Gamma, G)$ is finite, the space $\mathcal{X}(\Gamma, G)$ has finitely many components, because the real algebraic group $\text{Aut}(G)$ has only finitely many components.

Corollary H. *Let G be a simply connected solvable Lie group which is unipotently connected. Then for every Zariski-dense lattice Γ of G the space of lattice embeddings $\mathcal{X}(\Gamma, G)$ has finitely many connected components.*

The lattice Γ is said to be *locally rigid* in G if the $\text{Aut}(G)$ -orbit of the identity map $\varphi_0 = \text{id}_\Gamma: \Gamma \hookrightarrow G$ is open in $\mathcal{X}(\Gamma, G)$; see [20, I, Chap. 1, §6.1]. The lattice Γ is *deformation rigid* in G if the identity

component $\text{Aut}(G)_\circ$ acts transitively on the component of $\mathcal{X}(\Gamma, G)$ containing φ_0 ; see [26, §7]. In principle, Γ could be locally rigid without being deformation rigid.

The real algebraic hull A_Γ of Γ is a real algebraic group, associated to Γ in a functorial way, which we use to control the Zariski-dense lattice embeddings of Γ . Its construction is originally due to Mostow [19]. In Section 8.3 we establish a one-to-one correspondence between the structure set $\mathcal{S}^Z(\Gamma)$ and a certain collection $\mathcal{G}(\Gamma)$ of closed Lie subgroups of A_Γ . As explained in Section 9, this allows us to transfer the natural topology on $\mathcal{G}(\Gamma)$ induced by the Chabauty topology on the set of closed subgroups of A_Γ to $\mathcal{S}^Z(\Gamma)$. Using Theorem E and the result of Wang [27], we prove the following.

Theorem I. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . Then the structure set $\mathcal{S}^Z(\Gamma)$ has the discrete topology and the embedding of $\mathcal{D}(\Gamma, G)$ into $\mathcal{S}^Z(\Gamma)$ is continuous.*

In particular, every Zariski-dense lattice in a simply connected, solvable Lie group is locally rigid. In fact, from our proof of Theorem I we derive also the following corollary, for which a proof sketch is provided in [26, Proposition 7.2 and its Corollary 7.3].

Corollary J. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . Then Γ is deformation rigid in G .*

We conclude the section with a brief description of the organisation of the paper and a summary of the basic notation and terminology employed.

Organisation. In Section 2 we present a collection of instructive examples, illustrating our main results. In Section 3 we summarise important facts about lattices in solvable Lie groups and introduce tools with which to study them. In particular, we recall the algebraic hull construction for polycyclic groups and solvable Lie groups; after this we discuss the notion of tight Lie subgroups. In Section 4 we strengthen Mostow's theorem about the intersection of a lattice Γ with the nilradical of its ambient Lie group G . Section 5 is concerned with characterisations of unipotently connected Lie groups. In Section 6 we introduce and describe the space $\mathcal{G}(\Gamma)$ which assists in the description of Zariski-dense embeddings of a lattice Γ . Section 7 is devoted to the space $\mathcal{G}_{\Gamma, G}$ which gives a fibration for the deformation space $\mathcal{D}(\Gamma, G)$ of a Zariski-dense lattice Γ in G . Section 8 contains the proofs of most of our main results, namely all results from Theorem A up to Corollary H. Finally, in Section 9 we discuss the topologies on the structure set $\mathcal{S}^Z(\Gamma)$ and the deformation space $\mathcal{D}(\Gamma, G)$, proving Theorem I and its Corollary J.

Notation and terminology.

Polycyclic groups. A group Γ is *polycyclic* if there exists a finite series of subgroups $\Gamma = \Gamma_1 \supseteq \Gamma_2 \supseteq \dots \supseteq \Gamma_{r+1} = 1$ such that each quotient Γ_i/Γ_{i+1} is cyclic. The *rank* $\text{rk}(\Gamma)$ of a polycyclic group Γ is the number of infinite cyclic factors in such a series, which is an invariant of Γ . The *Fitting subgroup* $\text{Fitt}(\Gamma)$ of Γ is the maximal nilpotent normal subgroup of Γ . See [24] for further results on polycyclic groups.

Linear algebraic groups Let \mathbf{G} be a linear algebraic group, defined over a field of characteristic 0. If H is a subgroup of \mathbf{G} , we write \overline{H}^z for the *Zariski-closure* of H in \mathbf{G} and we denote by $u(H)$ the *collection of unipotent elements* in H . If H is solvable, $u(H)$ is a subgroup. The *identity component* of \mathbf{G} is denoted by \mathbf{G}° . The *unipotent radical* of \mathbf{G} , i.e., the maximal connected unipotent normal subgroup of \mathbf{G} , is denoted by $\text{Rad}_u(\mathbf{G})$. If \mathbf{G}° is a solvable group, then $\text{Rad}_u(\mathbf{G}) = u(\mathbf{G}) = u(\mathbf{G}^\circ)$, and $[\mathbf{G}^\circ, \mathbf{G}^\circ] \subseteq \text{Rad}_u(\mathbf{G})$.

The group \mathbf{G} has a *strong unipotent radical* if the centraliser of the unipotent radical $C_{\mathbf{G}}(\text{Rad}_u(\mathbf{G}))$ is contained in $\text{Rad}_u(\mathbf{G})$. If \mathbf{G} is defined over a field k , then we denote by $\text{Aut}_k(\mathbf{G})$ the group of k -defined automorphisms of \mathbf{G} . If \mathbf{G} has a strong unipotent radical and is defined over k , then the group $\text{Aut}_k(\mathbf{G})$ can be regarded as the group of k -points of a k -defined linear algebraic group; see [4]. See [5] for a general reference on linear algebraic groups.

Lie groups and lattices In the present paper all Lie groups are real Lie groups. Every simply connected Lie group is connected.

A *Lie subgroup* is an immersed submanifold which inherits a Lie group structure from the ambient group. The *identity component* of a Lie group G is denoted by G_\circ . The *nilradical* $\text{Nil}(G)$ of a Lie group G is the maximal connected nilpotent normal subgroup of G .

A real linear algebraic group A is the group of \mathbb{R} -points $A = \mathbf{A}_\mathbb{R}$ of a linear algebraic group \mathbf{A} defined over \mathbb{R} . In this case we write $A^\circ = \mathbf{A}^\circ_\mathbb{R}$. Such a group is also a Lie group with respect to its natural Euclidean (Hausdorff) topology. We have $A_\circ \leq A^\circ$ and $|A : A_\circ| < \infty$; see [18, Appendix] or, more generally, [29].

A closed subgroup H of a Lie group G is said to be *cocompact* (or *uniform*) if the quotient G/H is compact. A *lattice* in G is a discrete subgroup of finite co-volume, i.e. a discrete subgroup Γ such that the space G/Γ admits a G -invariant probability measure. If G is a solvable Lie group, then a subgroup Γ is a lattice in G if and only if Γ is discrete and cocompact. Let \mathfrak{g} be the Lie algebra associated to G , and let $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ denote the adjoint representation of G . We say that a lattice Γ is *Zariski-dense* in G if $\overline{\text{Ad}(\Gamma)}^z = \overline{\text{Ad}(G)}^z$ in the ambient real algebraic group $\text{GL}(\mathfrak{g})$. Every simply connected, nilpotent Lie group G admits naturally the structure of a unipotent real linear algebraic group by using exponential coordinates. With

respect to this real algebraic structure, a subgroup Γ is Zariski-dense in G if and only if the closure of Γ in the Euclidean topology of G is a cocompact subgroup. Every connected Lie subgroup of a unipotent real algebraic group is Zariski-closed. See [22] for a general reference on lattices of Lie groups.

2. INSTRUCTIVE EXAMPLES

The purpose of this section is to present a number of concrete examples and useful constructions of lattices in solvable Lie groups. While some of them are certainly well known, others highlight new insights. Many examples of rigid and non-rigid lattices in solvable Lie groups can be found in [26].

In order to describe some explicit Lie groups and lattices we parametrise 2-by-2 rotation matrices over \mathbb{R} by setting

$$(2.1) \quad R(t) = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix},$$

and we denote block diagonal matrices with blocks B_1, \dots, B_k , say, by $\text{diag}(B_1, \dots, B_k)$.

2.1. Uncountable deformation spaces. We provide a simple example of a non-Zariski-dense lattice Γ in a non-unipotently connected group G such that the deformation space $\mathcal{D}(\Gamma, G)$ is uncountable. We also explain that a construction of Milovanov, discussed in [26, Example 2.9], yields an example of a non-Zariski-dense lattice Γ in a unipotently connected group G such that $\mathcal{D}(\Gamma, G)$ is uncountable. The latter shows that the assumption of Zariski-denseness of Γ in Theorem A is not redundant.

Example 2.1. Consider the group $\widetilde{E(2)^+}$, the universal cover of the group of orientation-preserving isometries of the Euclidean plane. We realise an isomorphic copy of this group as follows:

$$G = V.X(\mathbb{R}) \cong \mathbb{R}^2 \rtimes \mathbb{R},$$

where $V = \mathbb{R} + \mathbb{R}i = \mathbb{C}$ and the one-parameter group $X(t)$ acts as multiplication by $e^{2\pi ti}$ on V . Then $\Gamma = (\mathbb{Z} + \mathbb{Z}i).X(\mathbb{Z}) \cong \mathbb{Z}^3$ is a lattice in G which is not Zariski-dense. We claim that the deformation space $\mathcal{D}(\Gamma, G)$ is uncountable.

We observe that $Z(G) = X(\mathbb{Z})$ and $\text{Nil}(G) = V$; in particular, G is not unipotently connected. If Λ is any lattice in the vector group V , then $\Lambda X(\mathbb{Z})$ is a lattice in G which is isomorphic to Γ . Now V is characteristic in G . Hence $\text{Aut}(G)$ acts on the collection of all lattices in V , and $\mathcal{D}(\Gamma, G)$ maps onto

$$\text{Aut}(G) \backslash \{\Lambda \mid \Lambda \text{ a lattice in } V\}.$$

Consequently, it suffices to show that the latter space is uncountable. For our purposes it is more convenient to work with the identity component $\text{Aut}(G)_\circ$ of $\text{Aut}(G)$; this is permissible, because $\text{Aut}(G)_\circ$ has finite index in $\text{Aut}(G)$. Two elements of V are conjugate in G if and only if they have the same modulus (as complex numbers). This shows that any automorphism of G acts on V as multiplication by a non-zero complex number, possibly followed by complex conjugation. It is easy to write down automorphisms of G which induce this action, and we deduce that the action of $\text{Aut}(G)_\circ$ on V is equivalent to the action of \mathbb{C}^* on \mathbb{C} by multiplication. The situation is now classical: the space of lattices in \mathbb{C} up to the action of \mathbb{C}^* is isomorphic to the upper half plane modulo the action of $\text{SL}_2(\mathbb{Z})$, hence uncountable.

Example 2.2. Based on an example of Milovanov and Starkov [26, Example 2.9], we construct a non-Zariski-dense lattice Γ in a unipotently connected group G such that $\mathcal{D}(\Gamma, G)$ is uncountable.

Let $A = \text{diag}(A_0, A_0) \in \text{GL}_4(\mathbb{Z})$, where $A_0 \in \text{GL}_2(\mathbb{Z})$ is the companion matrix of the polynomial $f = x^2 - 3x + 1$. Observe that f splits over \mathbb{R} into a product of two distinct linear factors $x - \lambda$ and $x - \lambda^{-1}$, say.

We define in $\text{GL}_4(\mathbb{R})$ the one-parameter subgroup

$$X(t) = \text{diag}(\lambda^t R(t), \lambda^{-t}, \lambda^{-t}) \quad (t \in \mathbb{R}).$$

We view A as an operator on a 4-dimensional vector space V over \mathbb{R} with basis v_1, v_2, v_3, v_4 , say. Thus $\Lambda = \bigoplus_{j=1}^4 \mathbb{Z}v_j$ is an A -invariant full \mathbb{Z} -lattice in V . Then V decomposes into a direct sum $V = V^\lambda \oplus V^{\lambda^{-1}}$ of A -invariant 2-dimensional subspaces corresponding to the eigenvalues of A . Choosing bases for these subspaces, we obtain a new basis for V so that the action of A on V with respect to this basis is given by $X(1)$. Then $\Gamma = \Lambda \rtimes X(\mathbb{Z})$ is a lattice of the simply connected, solvable Lie group $G = V \rtimes X(\mathbb{R})$. Observe that the maximal nilpotent normal subgroup of G is equal to $\text{Nil}(G) = V$ so that G is unipotently connected. Furthermore, the lattice Γ is not Zariski-dense in G .

In order to show that $\mathcal{D}(\Gamma, G)$ is uncountable we argue similarly as in Example 2.1. We consider the subspace of lattice embeddings of Γ into G which map $X(1)$ to itself. Such embeddings are in one-to-one correspondence with $X(\mathbb{Z})$ -equivariant embeddings of Λ into V , or in other words with elements of the centraliser of $X(\mathbb{Z})$ in $\text{GL}_4(\mathbb{R})$. This centraliser is isomorphic to $\text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R})$, preserving the subspaces V^λ and $V^{\lambda^{-1}}$. On the other hand we consider lattice embeddings of Γ into G mapping $X(1)$ to itself which are induced by automorphisms of G . These correspond to embeddings of Λ into V which are $X(\mathbb{R})$ -equivariant, or in other words to elements of the centraliser of $X(\mathbb{R})$ in $\text{GL}_4(\mathbb{R})$. They, too, preserve the subspaces V^λ and $V^{\lambda^{-1}}$,

but form a strictly smaller group isomorphic to $\mathbb{C}^* \times \mathrm{GL}_2(\mathbb{R})$. Similarly as in Example 2.1, we conclude that a subset of $\mathcal{D}(\Gamma, G)$ maps onto $\mathbb{C}^* \setminus \mathrm{GL}_2(\mathbb{R})$. The latter space is homeomorphic to the upper half plane. Thus $\mathcal{D}(\Gamma, G)$ is uncountable.

2.2. Finite deformation spaces. In view of Theorem A it is natural to try to construct Zariski-dense lattices in unipotently connected groups yielding deformation spaces of various finite cardinalities. We show that deformation spaces of arbitrarily large sizes can be obtained.

Example 2.3. Let $n \in \mathbb{N}_0$. Based on examples of Auslander and Starkov, we construct a Zariski-dense lattice Γ in a simply connected, solvable Lie group G such that the corresponding deformation space $\mathcal{D}(\Gamma, G)$ is finite of size $(n+1)!$.

Let $A \in \mathrm{GL}_4(\mathbb{Z})$ be the companion matrix of the polynomial $f = x^4 - 8x^3 + 10x^2 - 8x + 1$. It is easy to check that f splits over \mathbb{R} into a product of two linear factors and one irreducible quadratic factor. Indeed, the eigenvalues of A are

$$\lambda, \quad \lambda^{-1}, \quad \alpha = e^{2\pi i \vartheta}, \quad \bar{\alpha} = e^{-2\pi i \vartheta},$$

where $\lambda \in \mathbb{R}$ satisfies $\lambda > 1$ and $\vartheta \in (0, 1/2)$ is irrational.

Let $k \in \{0, \dots, n\}$. We define in $\mathrm{GL}_4(\mathbb{R})$ the one-parameter subgroup

$$X_k(t) = \mathrm{diag}(\lambda^t, \lambda^{-t}, R(t(\vartheta + k))) \quad (t \in \mathbb{R}),$$

where we use the notation introduced in (2.1). We view A as an operator on a 4-dimensional vector space V_k over \mathbb{R} with basis $v_{k,1}, v_{k,2}, v_{k,3}, v_{k,4}$, say. Thus $\Lambda_k = \bigoplus_{j=1}^4 \mathbb{Z}v_{k,j}$ is an A -invariant full \mathbb{Z} -lattice in V_k . Then V_k decomposes into a direct sum $V_k = V_k^\lambda \oplus V_k^{\lambda^{-1}} \oplus V_k^{\alpha, \alpha^{-1}}$ of A -invariant subspaces corresponding to the eigenvalues of A . Choosing appropriate bases for these subspaces, we obtain a new basis for V_k so that the action of A on V_k with respect to this basis is given by $X_k(1)$. Then $\Gamma_k = \Lambda_k \rtimes X_k(\mathbb{Z})$ is a lattice of the simply connected, solvable Lie group $G_k = V_k \rtimes X_k(\mathbb{R})$. According to [26, Example 4.1], the lattice Γ_k is Zariski-dense and rigid in G_k : every lattice embedding of Γ_k into G_k extends to an automorphism of G_k .

Next we define

$$G = \prod_{j=0}^n G_j \quad \text{and} \quad \Gamma = \prod_{j=0}^n \Gamma_j.$$

Clearly, Γ is a Zariski-dense lattice of the simply connected, solvable Lie group G . Whereas the Lie groups G_0, \dots, G_n are mutually non-isomorphic, their lattices $\Gamma_0, \dots, \Gamma_n$ are all isomorphic to one another. Thus we obtain an embedding

$$\mathrm{Sym}(n+1) \hookrightarrow \mathcal{X}(\Gamma, G), \quad \sigma \mapsto \varphi_\sigma$$

as follows. Viewing the symmetric group $\text{Sym}(n+1)$ as a permutation group of $\{0, \dots, n\}$, acting from the right, the image $\varphi_\sigma \in \mathcal{X}(\Gamma, G)$ of $\sigma \in \text{Sym}(n+1)$ is defined as

$$\varphi_\sigma: \Gamma \hookrightarrow G, \quad (g_j)_{j=0}^n \mapsto (\iota_{j\sigma, j}(g_{j\sigma}))_{j=0}^n,$$

where $\iota_{j\sigma, j}: \Gamma_{j\sigma} \rightarrow \Gamma_j$ denotes, for each $j \in \{0, \dots, n\}$, the isomorphism determined by $\iota_{j\sigma, j}(X_{j\sigma}(1)) = X_j(1)$ and $\iota_{j\sigma, j}(v_{j\sigma}^m) = v_j^m$ for $m \in \{1, 2, 3, 4\}$. We now claim that

$$(2.2) \quad \text{Aut}(G) = \prod_{j=0}^n \text{Aut}(G_j)$$

and that

$$(2.3) \quad \mathcal{D}(\Gamma, G) = \{[\varphi_\sigma] \mid \sigma \in \text{Sym}(n+1)\},$$

implying that the deformation space has size $(n+1)!$, as wanted.

In order to prove (2.2), consider $\psi \in \text{Aut}(G)$. We note that $V = \prod_{j=0}^n V_j = \text{Nil}(G)$ is mapped to itself by ψ . Furthermore, the factors V_j in this product are characterised as the G -invariant subgroups of V which are maximal with the property that their centralisers in G have codimension 1. Thus they are permuted among themselves by ψ . Moreover, the action of G on each group V_j by conjugation factors through $X_j(\mathbb{R})$ and thus defines G_j . As these Lie groups are mutually non-isomorphic, it is clear that ψ maps each V_j isomorphically onto itself. Now fix $k \in \{0, \dots, n\}$. We have $VX_k(\mathbb{R}) = C_G(\oplus_{j \neq k} V_j)$ and hence $V\psi(X_k(\mathbb{R})) = VX_k(\mathbb{R})$. Since $\psi(X_k(\mathbb{R}))$ and $\psi(X_j(\mathbb{R}))$ are to commute for any $j \in \{0, \dots, n\}$, we conclude that $\psi(X_k(\mathbb{R})) \subseteq V_k X_k(\mathbb{R}) = G_k$ and hence $\psi(G_k) = G_k$. This finishes the proof of (2.2).

It remains to justify the inclusion ‘ \subseteq ’ in (2.3). Let $\varphi: \Gamma \hookrightarrow G$ be a lattice embedding, and observe that $\varphi(\Gamma)$ is also Zariski-dense in G ; see Corollary 3.12. Set $\Lambda = \prod_{j=0}^n \Lambda_j$, the Fitting subgroup of Γ . By a theorem of Mostow [18, §5], the intersection $\varphi(\Gamma) \cap V$ is a lattice of $V \cong \mathbb{R}^{4(n+1)}$ and thus $\varphi(\Gamma)V/V$ is a lattice of $G/V \cong \mathbb{R}^n$. Since V is the maximal nilpotent normal subgroup of G , we must have $\varphi(\Gamma) \cap V = \varphi(\Lambda)$. Fix $k \in \{0, \dots, n\}$. The Zariski-closure $\overline{\varphi(\Lambda_k)}^Z$ of $\varphi(\Lambda_k)$ in V has dimension 4, and since the centraliser of Λ_k in Γ has co-rank 1, the centraliser of $\overline{\varphi(\Lambda_k)}^Z$ in G has co-dimension 1. As observed earlier, this implies that $\overline{\varphi(\Lambda_k)}^Z = V_{k\sigma}$, where $\sigma \in \text{Sym}(n+1)$ is a permutation of $\{0, \dots, n\}$. Arguing similarly as before, we deduce from $\Lambda X_k(\mathbb{Z}) = C_\Gamma(\oplus_{j \neq k} \Lambda_j)$ that

$$\overline{V\varphi(X_k(\mathbb{Z}))}^Z = C_G(\oplus_{j \neq k} \overline{\varphi(\Lambda_j)}^Z) = C_G(\oplus_{j \neq k} V_{j\sigma}) = VX_{k\sigma}(\mathbb{R}).$$

Since $\varphi(X_k(\mathbb{Z}))$ and $\varphi(X_j(\mathbb{Z}))$ are to commute for any $j \in \{0, \dots, n\}$, we conclude that $\varphi(X_k(\mathbb{Z})) \subseteq V_{k\sigma} X_{k\sigma}(\mathbb{R}) = G_{k\sigma}$, and hence $\varphi(\Gamma_k)$ is a lattice in $G_{k\sigma}$. Since $\varphi(\Gamma_k) \cong \Gamma_{k\sigma}$ and since $\Gamma_{k\sigma} = \varphi_{\sigma^{-1}}(\Gamma_k)$ is rigid in

$G_{k\sigma}$ we find $\gamma_{k\sigma} \in \text{Aut}(G_{k\sigma})$ such that $\gamma_{k\sigma} \circ \varphi|_{\Gamma_k} = \varphi_{\sigma^{-1}}|_{\Gamma_k}$. In view of (2.2), this proves (2.3).

2.3. Countably infinite deformation spaces. Corollary F states that the deformation space of a Zariski-dense lattice in a simply connected, solvable Lie group G is at most countably infinite. We construct a Zariski-dense lattice in a non-unipotently connected group which has an infinite deformation space. The example illustrates that the assumption of unipotent connectedness of G in Theorem A is not redundant.

Lemma 2.4. *There are infinitely many polynomials $f \in \mathbb{Z}[x]$ which factorise over \mathbb{C} as*

$$f = (x - \alpha)(x - \bar{\alpha})(x - \alpha^{-1})(x - \bar{\alpha}^{-1}),$$

where $\alpha = \lambda e^{2\pi i \vartheta}$ with $\lambda \in \mathbb{R}_{>1}$ and $\vartheta \in (0, 1/2)$ irrational. We can further arrange that the polynomials are irreducible over \mathbb{Q} .

Proof. We consider the collection $\mathcal{F} \subseteq \mathbb{Z}[x]$ of all polynomials

$$x^4 - ax^3 + bx^2 - ax + 1$$

with $a, b \in \mathbb{N}$ such that $a, b \equiv_2 1$ and $b > \max\{a^2/2, 2a + 2\}$.

Since $x^4 + x^3 + x^2 + x + 1$ is irreducible over the field \mathbb{F}_2 of cardinality 2, we conclude that all polynomials in \mathcal{F} are irreducible over \mathbb{Z} , and hence over \mathbb{Q} .

Now consider $f = x^4 - ax^3 + bx^2 - ax + 1 \in \mathcal{F}$. Two short computations show that

- (i) f has no real roots, because $b > a^2/2$.
- (ii) f has no complex roots of modulus 1, because $b > 2a + 2$.

Hence, $f = (x - \alpha)(x - \bar{\alpha})(x - \alpha^{-1})(x - \bar{\alpha}^{-1})$, where $\alpha = \lambda e^{2\pi i \vartheta}$ with $\lambda \in \mathbb{R}_{>1}$ and $\vartheta \in (0, 1/2)$. From $a = \alpha + \bar{\alpha} + \alpha^{-1} + \bar{\alpha}^{-1} = (\lambda + \lambda^{-1})2 \cos(2\pi \vartheta)$ we see that a and ϑ determine λ , and hence b . Consequently, if we fix a , then different values for b correspond to different values for ϑ . On the other hand, the degree of the field $\mathbb{Q}(\alpha, \bar{\alpha})$ over \mathbb{Q} is uniformly bounded by 8 and thus $\mathbb{Q}(\alpha, \bar{\alpha})$ contains roots of unity only up to a certain, uniformly bounded order. Since $e^{4\pi i \vartheta} = \alpha \bar{\alpha}^{-1} \in \mathbb{Q}(\alpha, \bar{\alpha})$, this means that, for fixed a , for almost all b , the resulting value for ϑ must be irrational.

Hence, for fixed a , for almost all b the polynomial $f = x^4 - ax^3 + bx^2 - ax + 1 \in \mathcal{F}$ has the required properties \square

Example 2.5. We construct a Zariski-dense lattice Γ in a simply connected, solvable Lie group G such that $\Gamma/\text{Fitt}(\Gamma)$ is not torsion-free. Consequently, the Lie group G is not unipotently connected; see Lemma 5.2. We go on to show that the deformation space $\mathcal{D}(\Gamma, G)$ is infinite.

Let $A \in \text{GL}_4(\mathbb{Z})$ with complex eigenvalues

$$\alpha = \lambda e^{2\pi i \vartheta}, \quad \bar{\alpha} = \lambda e^{-2\pi i \vartheta}, \quad \bar{\alpha}^{-1} = \lambda^{-1} e^{2\pi i \vartheta}, \quad \alpha^{-1} = \lambda^{-1} e^{-2\pi i \vartheta},$$

where $\lambda \in \mathbb{R}$ satisfies $\lambda > 1$ and $\vartheta \in (0, 1/2)$ is irrational. For instance, one can take the companion matrix of one of the polynomials in Lemma 2.4.

Alternatively, one can take for A the companion matrix of the concrete polynomial $f = x^4 + 22x^3 + 150x^2 + 22x + 1$. One checks that $A = B^3$ for $B \in \mathrm{GL}_4(\mathbb{R})$ with characteristic polynomial

$$x^4 + 4x^3 + 3x^2 - 2x + 1 = (x + 1)^4 - 3(x + 1)^2 + 3.$$

This polynomial appears in a construction of Wilking; see [30, Example 2.1]. He showed that the eigenvalues of B , which one computes easily, have angular component $2\pi i\vartheta$ with irrational ϑ . Thus A has the desired property.

We define in $\mathrm{GL}_5(\mathbb{R})$ the one-parameter subgroups

$$\begin{aligned} X(t) &= \mathrm{diag}(\lambda^t R(t\vartheta), \lambda^{-t} R(-t\vartheta), 1) & (t \in \mathbb{R}), \\ Y(t) &= \mathrm{diag}(2^t R(t/2), 2^t R(-t/2), 2^t) & (t \in \mathbb{R}), \end{aligned}$$

where we employ the notation introduced in (2.1). Clearly, $X(\mathbb{R})$ and $Y(\mathbb{R})$ commute with one another.

We view A as an operator on a 4-dimensional vector space V over \mathbb{R} with basis v_1, v_2, v_3, v_4 , say. Thus $\Lambda = \bigoplus_{i=1}^4 \mathbb{Z}v_i$ is an A -invariant full \mathbb{Z} -lattice in V . Then V decomposes into a direct sum $V = V_{\alpha, \bar{\alpha}} \oplus V_{\alpha^{-1}, \bar{\alpha}^{-1}}$ of A -invariant planes corresponding to the eigenvalue pairs $\alpha, \bar{\alpha}$ and $\alpha^{-1}, \bar{\alpha}^{-1}$. Choosing appropriate bases for $V_{\alpha, \bar{\alpha}}$ and $V_{\alpha^{-1}, \bar{\alpha}^{-1}}$, we obtain a new basis e_1, e_2, e_3, e_4 so that, if the abelian group V is embedded into $\mathrm{GL}_5(\mathbb{R})$ via

$$\eta: V \rightarrow \mathrm{GL}_5(\mathbb{R}), \quad \sum_{i=1}^4 x_i e_i \mapsto \begin{pmatrix} \mathrm{Id} & \underline{x} \\ 0 & 1 \end{pmatrix},$$

where $\underline{x} = (x_1, x_2, x_3, x_4)^t$, then the original action of A on V is isomorphic to that of $X(1)$ on $\eta(V)$ by conjugation in $\mathrm{GL}_5(\mathbb{R})$.

Now we consider the simply connected, solvable Lie group

$$G = \eta(V).X(\mathbb{R})Y(\mathbb{R}) \cong \mathbb{R}^4 \rtimes (\mathbb{R} \times \mathbb{R})$$

and its lattice

$$\Gamma = \eta(\Lambda).X(\mathbb{Z})Y(\mathbb{Z}) \cong \mathbb{Z}^4 \rtimes (\mathbb{Z} \times \mathbb{Z}).$$

It is easily seen that $Z(G) = Y(2\mathbb{Z})$ and that $VY(2\mathbb{Z})$ is the maximal nilpotent normal subgroup of G . Thus $\mathrm{Nil}(G) = V$ and G is not unipotently connected. Furthermore, we have

$$\mathrm{Fitt}(\Gamma) = \eta(\Lambda).Y(2\mathbb{Z}) \cong \mathbb{Z}^5 \quad \text{and} \quad \Gamma / \mathrm{Fitt}(\Gamma) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Next we prove that Γ is Zariski-dense in G . For this it suffices to show that Γ and G have the same Zariski-closure in $\mathrm{GL}_5(\mathbb{R})$. Clearly,

$\eta(V)$ is Zariski-closed and $\overline{\eta(\Lambda)}^z = \overline{\eta(V)}^z = \eta(V)$. The group $X(\mathbb{R})$ is contained in the 2-dimensional real algebraic torus

$$T = T_s \times T_c,$$

where

$$\begin{aligned} T_s &= \{\text{diag}(r, r, r^{-1}, r^{-1}, 1) \mid r \in \mathbb{R}^*\} \cong \text{GL}_1(\mathbb{R}), \\ T_c &= \{\text{diag}(R(r), R(-r), 1) \mid r \in \mathbb{R}\} \cong \text{SO}_2(\mathbb{R}). \end{aligned}$$

From the eigenvalues of the elements in $X(\mathbb{Z}) \subseteq X(\mathbb{R})$ we see that $X(\mathbb{Z})$ is neither contained in a split real algebraic torus nor in a compact real algebraic torus. This means that the algebraic closure of $X(\mathbb{Z})$ is at least 2-dimensional and hence $\overline{X(\mathbb{Z})}^z = \overline{X(\mathbb{R})}^z = T$. Finally, we note that the group $Y(\mathbb{R})$ is contained in the 2-dimensional real algebraic torus

$$S = S_s \times S_c,$$

where

$$S_s = \{\text{diag}(r, r, r, r, r) \mid r \in \mathbb{R}^*\} \cong \text{GL}_1(\mathbb{R}) \quad \text{and} \quad S_c = T_c.$$

Again from the eigenvalues of elements in $Y(\mathbb{R})$ it follows that $\overline{Y(\mathbb{R})}^z = S$, and a small computation shows that

$$\overline{Y(\mathbb{Z})}^z = S_s \times \langle \text{diag}(-1, -1, -1, -1, 1) \rangle.$$

Altogether we see that

$$\overline{\Gamma}^z = \eta(V)(T_s \times S_s \times T_c) = \overline{G}^z.$$

It remains to be shown that $\mathcal{D}(\Gamma, G)$ is infinite, and hence countably infinite by Corollary F. The automorphism group $\text{Aut}(G)$ acts on $G/\text{Nil}(G) = G/V$ as a finite group of automorphisms; see Lemma 6.9. On the other hand

$$\Gamma_m = \eta(\Lambda).X(\mathbb{Z})Y((2m+1)\mathbb{Z}), \quad m \in \mathbb{N},$$

gives a countably infinite family of lattices in G , which are distinct modulo V and each of which is isomorphic to Γ . Thus $\mathcal{D}(\Gamma, G)$ is infinite.

2.4. Unipotently connected groups and their lattices. The construction in Example 2.5 also provides interesting examples of lattices in unipotently connected groups. We comment on this matter just after Lemma 5.2.

3. ALGEBRAIC HULLS, DENSITY PROPERTIES OF LATTICES AND TIGHT LIE SUBGROUPS

In this section we develop important facts about lattices in solvable Lie groups. In particular, we introduce the algebraic hull construction, our central tool in the study of such lattices. Most of the material is

implicit in the works of Mostow, in particular [17, 18, 19], and Auslander [1]; see also [22] and [26]. In the last part of the section we define and parametrise tight Lie subgroups of a solvable real algebraic group.

3.1. Algebraic hulls. One of our key tools is the algebraic hull construction for polycyclic groups and solvable Lie groups which is originally due to Mostow [19]. In the context of polycyclic groups, the algebraic hull can be regarded as a generalisation of the Mal'tsev completion of a finitely generated, torsion-free nilpotent group. The construction is related to the notion of semisimple splittings, which originates in the works of Mal'tsev, Wang and Auslander on solvmanifolds. We recall some of the key features of the algebraic hull construction. For further details see [22, Chap. IV], [2, App. A] and [3, Chap. 1].

3.1.1. Polycyclic groups. Let Γ be a polycyclic group. It is known that $C_\Gamma(\text{Fitt}(\Gamma)) \subseteq \text{Fitt}(\Gamma)$; see [24, §2B]. Moreover, the following conditions are equivalent:

- $\text{Fitt}(\Gamma)$ is torsion-free;
- besides the trivial group, Γ has no finite normal subgroups.

Suppose that one of these conditions is satisfied. Then there exist a \mathbb{Q} -defined linear algebraic group \mathbf{A} and an embedding $\iota: \Gamma \hookrightarrow \mathbf{A}$ such that $\iota(\Gamma) \subseteq \mathbf{A}_{\mathbb{Q}}$ and

- (i) $\iota(\Gamma)$ is Zariski-dense in \mathbf{A} ,
- (ii) \mathbf{A} has a strong unipotent radical, i.e., $C_{\mathbf{A}}(\text{Rad}_u(\mathbf{A})) \subseteq \text{Rad}_u(\mathbf{A})$,
- (iii) $\dim \text{Rad}_u(\mathbf{A}) = \text{rk } \Gamma$.

Moreover, the construction $\iota: \Gamma \hookrightarrow \mathbf{A}$ is uniquely determined up to \mathbb{Q} -isomorphism of linear algebraic groups; see Corollary 3.2. We thus refer to \mathbf{A} , together with a possibly implicit embedding ι , as the *algebraic hull* of Γ . In dealing with several groups at the same time, it will be convenient to denote the algebraic hull of Γ by \mathbf{A}_Γ and to assume that ι is simply the inclusion map: $\Gamma \subseteq \mathbf{A}_\Gamma$.

Now let k be a field of characteristic 0. Then a *k-defined algebraic hull* for Γ is a k -defined linear algebraic group \mathbf{A} together with an embedding $\iota: \Gamma \hookrightarrow \mathbf{A}$ such that $\iota(\Gamma) \subseteq \mathbf{A}_k$ and satisfying conditions (i), (ii), (iii) above. The group of k -points $A = \mathbf{A}_k$, together with the embedding $\iota: \Gamma \hookrightarrow A$, is called a *k-algebraic hull* of Γ . In the special case $k = \mathbb{R}$, we call A the *real algebraic hull* of Γ . If k is an algebraic number field with ring of integers \mathcal{O} , then the k -defined algebraic hull \mathbf{A} of Γ has the additional property that

- (iv) for any k -defined representation $\varrho: \mathbf{A} \rightarrow \text{GL}_n(\mathbb{C})$, $n \in \mathbb{N}$, the pre-image $\iota^{-1}\varrho^{-1}(\text{GL}_n(\mathcal{O}))$ has finite index in Γ .

We remark that the algebraic hull \mathbf{A}_Γ of a polycyclic group Γ need not be connected, even if Γ is poly-(infinite cyclic). For instance, if $\Gamma \cong \mathbb{Z} \rtimes \mathbb{Z}$ is the fundamental group of the Klein bottle, then the real algebraic hull of Γ is isomorphic to $\mathbb{R}^2 \rtimes \{1, -1\}$.

The key property of the algebraic hull construction is recorded in the following lemma.

Lemma 3.1 (Extension Lemma). *Let Γ be a polycyclic group. Let k be a field of characteristic 0, and suppose that \mathbf{A} is a k -defined algebraic hull for Γ . Let \mathbf{B} be a linear algebraic k -group which has a strong unipotent radical. Then every homomorphism $\varphi: \Gamma \rightarrow \mathbf{B}$ with Zariski-dense image $\varphi(\Gamma) \subseteq \mathbf{B}$ extends uniquely to a k -defined morphism of algebraic groups $\Phi: \mathbf{A} \rightarrow \mathbf{B}$.*

Proof. A variant of this lemma is proved in [3, Proposition 1.4]; compare also [22, Lemma 4.41]. \square

Corollary 3.2. *Let Γ be a polycyclic group. Let k be a field of characteristic 0, and suppose that \mathbf{A}, \mathbf{B} are k -defined algebraic hulls for Γ . Then the identity map on Γ extends to a k -defined isomorphism of algebraic groups $\Phi: \mathbf{A} \rightarrow \mathbf{B}$.*

Proof. By Lemma 3.1, the identity map on Γ extends to k -defined morphisms of algebraic groups $\Phi: \mathbf{A} \rightarrow \mathbf{B}$ and $\Psi: \mathbf{B} \rightarrow \mathbf{A}$. Note that $\Psi \circ \Phi: \mathbf{A} \rightarrow \mathbf{A}$ restricts to the identity on Γ . Since Γ is Zariski-dense in \mathbf{A} , we conclude that $\Psi \circ \Phi = \text{id}_{\mathbf{A}}$. Similarly, one shows that $\Phi \circ \Psi = \text{id}_{\mathbf{B}}$. Hence Φ and Ψ are mutual inverses of each other, and $\Phi: \mathbf{A} \rightarrow \mathbf{B}$ is an isomorphism, as wanted. \square

3.1.2. *Simply connected, solvable Lie groups.* Likewise, every simply connected, solvable Lie group G admits an algebraic hull: there exist an \mathbb{R} -defined linear algebraic group \mathbf{A} and an injective Lie homomorphism $\iota: G \hookrightarrow \mathbf{A}$ such that $\iota(G) \subseteq \mathbf{A}_{\mathbb{R}}$ and

- (i)' $\iota(G)$ is Zariski-dense in \mathbf{A} ,
- (ii)' \mathbf{A} has a strong unipotent radical, i.e., $C_{\mathbf{A}}(\text{Rad}_{\mathbf{u}}(\mathbf{A})) \subseteq \text{Rad}_{\mathbf{u}}(\mathbf{A})$,
- (iii)' $\dim \text{Rad}_{\mathbf{u}}(\mathbf{A}) = \dim G$.

Again, the construction $\iota: G \hookrightarrow \mathbf{A}$ is uniquely determined up to \mathbb{R} -isomorphism of linear algebraic groups. We thus refer to \mathbf{A} , together with a possibly implicit embedding ι , as the *algebraic hull* of G . It will be convenient to denote the algebraic hull of G by \mathbf{A}_G and to assume that ι is simply the inclusion map: $G \subseteq \mathbf{A}_G$.

We observe that \mathbf{A}_G is a connected solvable group. A result similar to Lemma 3.1 holds for the algebraic hull \mathbf{A}_G of G . The real algebraic group $A = \mathbf{A}_{\mathbb{R}}$, together with the embedding $\iota: G \hookrightarrow A$, is called the *real algebraic hull* of G , and we denote it by A_G .

The following result (see [2, Proposition 2.3]) states, in particular, that G is a closed Lie subgroup in its real algebraic hull A_G .

Proposition 3.3. *Let G be a simply connected, solvable Lie group, and let $A = A_G$ be its real algebraic hull. Let U denote the maximal unipotent normal subgroup of A , and let T be a maximal reductive subgroup of A . Then $A = U \rtimes T$, and we denote by $\Upsilon: A \rightarrow U$,*

$g = ut \mapsto u$ the algebraic projection associated to the choice of T . Then G is a closed normal subgroup of A and $A = G \rtimes T$. Moreover, Υ restricts to a diffeomorphism $\Upsilon|_G: G \rightarrow U$.

3.2. Density properties of lattices. Let Γ be a lattice in a simply connected, solvable Lie group G . In this section we consider the Zariski-closure of Γ in the algebraic hull of G . Subsequently, we present several applications to Zariski-dense lattices, which are more closely linked to their ambient Lie groups than general lattices.

3.2.1. General lattices. We start with a simple criterion for recognising lattices in simply connected, solvable Lie groups.

Lemma 3.4. *Let Γ be a discrete subgroup of a simply connected, solvable Lie group G . Then Γ is poly-(infinite cyclic) and $\text{rk } \Gamma \leq \dim G$. Moreover, Γ is a lattice in G if and only if $\text{rk } \Gamma = \dim G$.*

Proof. The group $\Gamma \cap \text{Nil}(G)$ is discrete in $\text{Nil}(G)$ and therefore a finitely generated, torsion-free, nilpotent group. Moreover, $\Gamma/(\Gamma \cap \text{Nil}(G))$ embeds into the vector group $G/\text{Nil}(G)$ and is a finitely generated, torsion-free, abelian group. Thus Γ is poly-(infinite cyclic). The proof of [22, Proposition 3.7] shows that $\text{rk } \Gamma \leq \dim G$ and that equality holds if Γ is a lattice in G . Conversely, suppose that $\text{rk } \Gamma = \dim G$. We need to show that Γ is cocompact in G . For this it is enough to prove that the cohomological dimension $\text{cdim } \Gamma$ of Γ is equal to $\dim G$; see [6, §VIII.9.4]. This is true, because $\text{cdim } \Gamma = \text{rk } \Gamma$ by [9, §8.8]. \square

Now let Γ be a lattice in a simply connected, solvable Lie group G . Then the inclusion $G \subseteq \mathbf{A}_G$ of G into its algebraic hull \mathbf{A}_G restricts to an inclusion $\Gamma \subseteq \mathbf{A}_G$.

Lemma 3.5. *Let Γ be a lattice in a simply connected, solvable Lie group G . Then Γ is Zariski-dense in the unipotent part of \mathbf{A}_G :*

$$u(\overline{\Gamma}^z) = u(\mathbf{A}_G) = \text{Rad}_u(\mathbf{A}_G),$$

where $\overline{\Gamma}^z$ denotes the Zariski-closure of Γ in \mathbf{A}_G .

Proof. Since \mathbf{A}_G is solvable, $u(\mathbf{A}_G) = \text{Rad}_u(\mathbf{A}_G)$. Clearly, $u(\overline{\Gamma}^z)$ is contained in $u(\mathbf{A}_G)$. One of the properties of the algebraic hull is that $\dim u(\mathbf{A}_G) = \dim G$. Therefore, to show that $u(\overline{\Gamma}^z) = u(\mathbf{A}_G)$ it suffices to show that $\dim u(\overline{\Gamma}^z) = \dim G$.

Writing $A = \mathbf{A}_{\mathbb{R}}$ for the real algebraic hull of G , we have $A = U \rtimes T$, where $U = u(A)$ is the maximal unipotent normal subgroup of A and T is a maximal reductive subgroup. The semidirect product decomposition $A = U \rtimes T$ induces a natural action α of A on U :

$$(3.1) \quad \alpha(g) \cdot v = u {}^t v = utvt^{-1} \quad \text{for } g = ut \in U \rtimes T \text{ and } v \in U.$$

By Proposition 3.3, we have $A = G \rtimes T$ so that the action α in (3.1) restricts to a simply transitive action of G on U . Since $\Gamma \subseteq \overline{\Gamma}_{\mathbb{R}}^z$, the

Γ -orbit $\alpha(\Gamma) \cdot 1$ is contained in $u(\overline{\Gamma}^z)_{\mathbb{R}}$, and, in fact, the group $u(\overline{\Gamma}^z)_{\mathbb{R}}$ is invariant under the action of $\alpha(\Gamma)$. Since Γ acts properly discontinuously and freely on the contractible Lie group $u(\overline{\Gamma}^z)_{\mathbb{R}}$, the quotient $\Gamma \backslash u(\overline{\Gamma}^z)_{\mathbb{R}}$ is a compact manifold and an Eilenberg–MacLane space of type $K(\Gamma, 1)$. Since also G/Γ is a compact Eilenberg–MacLane space of type $K(\Gamma, 1)$, we deduce that $\dim u(\overline{\Gamma}^z) = \dim G$. \square

We remark that, with some extra work, the conclusion in Lemma 3.5 can be strengthened: Γ is Zariski-dense in the real split part of \mathbf{A}_G . Under weaker assumptions, we can formulate the following result.

Lemma 3.6. *Let Γ be a discrete subgroup of a simply connected, solvable Lie group G . Then $\dim u(\overline{\Gamma}^z) = \text{rk } \Gamma$, where $\overline{\Gamma}^z$ denotes the Zariski-closure of Γ in \mathbf{A}_G .*

Proof. The group Γ is polycyclic. Hence a lemma of Mostow (see [22, Lemma 4.36]) shows that $\dim u(\overline{\varrho(\Gamma)}^z)_{\mathbb{R}} \leq \text{rk } \Gamma$ for any linear representation $\varrho: \Gamma \rightarrow \text{GL}_n(\mathbb{R})$. Applying this to the inclusion $\Gamma \subseteq A_G$ of Γ into the real algebraic hull A_G of G , we deduce that $\dim u(\overline{\Gamma}^z)_{\mathbb{R}} \leq \text{rk } \Gamma$. The (last part of the) proof of Lemma 3.5 shows that Γ acts properly discontinuously and freely on the contractible Lie group $u(\overline{\Gamma}^z)_{\mathbb{R}}$. Therefore we have $\dim u(\overline{\Gamma}^z)_{\mathbb{R}} \geq \text{cdim } \Gamma$, where $\text{cdim } \Gamma$ denotes the cohomological dimension of Γ . By [9, §8.8] we have $\text{cdim } \Gamma = \text{rk } \Gamma$. Thus $\dim u(\overline{\Gamma}^z) = \dim u(\overline{\Gamma}^z)_{\mathbb{R}} = \text{rk } \Gamma$. \square

Proposition 3.7. *Let Γ be a lattice in a simply connected, solvable Lie group G . Let \mathbf{A}_{Γ} be an \mathbb{R} -defined algebraic hull of Γ , and let \mathbf{A}_G be an algebraic hull of G . Then the inclusion $\varphi: \Gamma \hookrightarrow G \subseteq \mathbf{A}_G$ extends uniquely to an \mathbb{R} -defined embedding of algebraic groups $\Phi: \mathbf{A}_{\Gamma} \hookrightarrow \mathbf{A}_G$, which restricts to an isomorphism of unipotent radicals.*

Proof. Considering Γ as a subgroup of \mathbf{A}_G , put $\mathbf{H} = \overline{\Gamma}^z$, the algebraic closure of Γ in \mathbf{A}_G . Since $\Gamma \subseteq (\mathbf{A}_G)_{\mathbb{R}}$, the group \mathbf{H} is defined over \mathbb{R} . By Lemma 3.5, the group \mathbf{H} has unipotent radical $\text{Rad}_u(\mathbf{H}) = \text{Rad}_u(\mathbf{A}_G)$. In particular, \mathbf{H} has a strong unipotent radical. Therefore, by Lemma 3.1, the inclusion $\Gamma \subseteq \mathbf{H}$ extends to a surjective \mathbb{R} -defined morphism $\Psi: \mathbf{A}_{\Gamma} \rightarrow \mathbf{H}$ of algebraic groups. Observe that $\dim \text{Rad}_u(\mathbf{A}_{\Gamma}) = \text{rk } \Gamma = \dim G = \dim \text{Rad}_u(\mathbf{H})$. Hence composing Ψ with the inclusion $\mathbf{H} \subseteq \mathbf{A}_G$ yields a morphism $\Phi: \mathbf{A}_{\Gamma} \rightarrow \mathbf{A}_G$ which restricts to an isomorphism $\text{Rad}_u(\mathbf{A}_{\Gamma}) \rightarrow \text{Rad}_u(\mathbf{H}) = \text{Rad}_u(\mathbf{A}_G)$ of the unipotent radicals. The kernel \mathbf{K} of Φ intersects $\text{Rad}_u(\mathbf{A}_{\Gamma})$ trivially and hence centralises $\text{Rad}_u(\mathbf{A}_{\Gamma})$. Since \mathbf{A}_{Γ} has a strong unipotent radical, we conclude that \mathbf{K} is trivial. \square

From $\text{rk } \Gamma = \dim G$ we deduce that \mathbf{A}_G is an algebraic hull for Γ if and only if Γ is Zariski-dense in \mathbf{A}_G . Of course, in general this need not be the case. For instance, Example 2.1 illustrates that the non-abelian Lie group $\widehat{E(2)^+}$ admits a host of abelian lattices.

3.2.2. *Zariski-dense lattices.* We continue our discussion with several applications to Zariski-dense lattices.

Lemma 3.8. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . Then*

- (1) $[\Gamma, \Gamma]$ is a lattice in the Lie group $[G, G]$;
- (2) $\Gamma \cap [G, G]$ is a lattice in the Lie group $[G, G]$;
- (3) The subgroup $\Gamma[G, G]$ is closed in G .

Proof. Let U be the maximal unipotent subgroup of $A_\Gamma = A_G$. Since $[G, G] \leq U$, it is a Zariski-closed subgroup of the real algebraic group A_Γ . Therefore, $[G, G] = [A_\Gamma, A_\Gamma]$, and each of the discrete subgroups $[\Gamma, \Gamma]$ and $\Gamma \cap [G, G]$ is Zariski-dense in the latter. Consequently, $[\Gamma, \Gamma]$ and $\Gamma \cap [G, G]$ are (cocompact) lattices in $[G, G]$. In particular, it follows that the image of $[G, G]$ in the Hausdorff space G/Γ is closed. Therefore, $\Gamma[G, G]$ is closed in G . \square

Next we characterise Zariski-dense lattices in terms of algebraic hulls.

Proposition 3.9. *Let Γ be a lattice in a simply connected, solvable Lie group G . Then Γ is Zariski-dense in G if and only if Γ is Zariski-dense subgroup of \mathbf{A}_G .*

Proof. Write $\mathbf{A} = \mathbf{A}_G$. Since G is simply connected, we have $\text{Ad}(G) \cong G/Z(G)$. Since we are working in characteristic 0, we have $\text{Ad}(\mathbf{A}) \cong \mathbf{A}/Z(\mathbf{A})$. By the algebraic hull construction, G is Zariski-dense in \mathbf{A} . Thus $Z(G) = G \cap Z(\mathbf{A})$ and $\text{Ad}(G)$ embeds as a Zariski-dense subgroup $GZ(\mathbf{A})/Z(\mathbf{A})$ into $\mathbf{A}/Z(\mathbf{A})$.

Thus Γ is Zariski-dense in G if and only if $\bar{\Gamma}^Z Z(\mathbf{A}) = \mathbf{A}$. Hence to prove our claim it suffices to show that $Z(\mathbf{A}) \subseteq \bar{\Gamma}^Z$. Since \mathbf{A} has a strong unipotent radical, we have $Z(\mathbf{A}) \subseteq \text{Rad}_u(\mathbf{A})$. Thus by Lemma 3.5 we have $Z(\mathbf{A}) \subseteq \bar{\Gamma}^Z$. \square

The next result shows, in particular, that extensions of automorphisms are unique.

Proposition 3.10. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G , and let $\varphi: \Gamma \rightarrow G$. If there exists $\Phi \in \text{Aut}(G)$ which extends φ , then Φ is unique.*

Proof. Assume that $\Phi_1, \Phi_2: G \rightarrow G$ are Lie automorphisms of G such that $\Phi_1|_\Gamma = \varphi = \Phi_2|_\Gamma$. Then by the Lie group version of Lemma 3.1 the maps Φ_1, Φ_2 extend to algebraic automorphisms $\tilde{\Phi}_1, \tilde{\Phi}_2$ of \mathbf{A}_G which coincide on the Zariski-dense subgroup Γ . It follows that $\Phi_1 = \Phi_2$. \square

If Γ is a Zariski-dense lattice in a simply connected, solvable Lie group G , then Γ is a torsion-free polycyclic group, and the next result shows that \mathbf{A}_G (respectively A_G) also constitutes an algebraic hull \mathbf{A}_Γ (respectively real algebraic hull A_Γ) of Γ .

Corollary 3.11. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . Then the following hold.*

- (1) *The inclusion $\Gamma \subseteq G$ extends uniquely to an \mathbb{R} -defined isomorphism of algebraic hulls $\mathbf{A}_\Gamma \rightarrow \mathbf{A}_G$.*
- (2) *The inclusion $\Gamma \subseteq \mathbf{A}_\Gamma$ extends uniquely to an inclusion homomorphism of Lie groups $G \hookrightarrow (\mathbf{A}_\Gamma)_\mathbb{R}$ and the image of G is closed in $(\mathbf{A}_\Gamma)_\mathbb{R}$.*

Proof. (1) By Proposition 3.7, the inclusion $\Gamma \subseteq G \subseteq \mathbf{A}_G$ extends uniquely to an \mathbb{R} -defined embedding $\Phi: \mathbf{A}_\Gamma \hookrightarrow \mathbf{A}_G$ of algebraic groups. Since Γ is Zariski-dense, $\Phi(\mathbf{A}_\Gamma) = \mathbf{A}_G$, and Φ is a bijection. Note that \mathbf{A}_G is an \mathbb{R} -defined algebraic hull for Γ . Thus Corollary 3.2 implies that Φ^{-1} is an \mathbb{R} -defined morphism, showing that Φ is an \mathbb{R} -defined isomorphism of algebraic hulls $\mathbf{A}_\Gamma \rightarrow \mathbf{A}_G$.

(2) This is a direct consequence of (1) and Proposition 3.3. \square

Let Γ be a lattice in a simply connected, solvable Lie group G . Recall that $\mathcal{X}(\Gamma, G)$ denotes the space of all lattice embeddings of Γ into G . We now consider, temporarily, the following subspace

$$\mathcal{X}^Z(\Gamma, G) = \{\varphi: \Gamma \hookrightarrow G \mid \varphi(\Gamma) \text{ is a Zariski-dense lattice in } G\}.$$

The next corollary shows that, if Γ can be embedded as a Zariski-dense lattice into G at all, then $\mathcal{X}^Z(\Gamma, G) = \mathcal{X}(\Gamma, G)$.

Corollary 3.12. *Let $\varphi_1, \varphi_2: \Gamma \hookrightarrow G$ be embeddings of Γ as a lattice into a simply connected, solvable Lie group G . If $\varphi_1(\Gamma)$ is Zariski-dense in G , then $\varphi_2(\Gamma)$ is also Zariski-dense in G .*

Proof. Suppose that $\varphi_1(\Gamma)$ is Zariski-dense. Let $\Phi_1, \Phi_2: \mathbf{A}_\Gamma \hookrightarrow \mathbf{A}_G$ denote the extensions of φ_1, φ_2 to the level of algebraic hulls, provided by Proposition 3.7. By Corollary 3.11, $\Phi_1: \mathbf{A}_\Gamma \rightarrow \mathbf{A}_G$ is an isomorphism. In particular, $\dim \mathbf{A}_\Gamma = \dim \mathbf{A}_G$, and this shows that the embedding $\Phi_2: \mathbf{A}_\Gamma \rightarrow \mathbf{A}_G$ is surjective. Since Γ is Zariski-dense in \mathbf{A}_Γ , the group $\varphi_2(\Gamma) = \Phi_2(\Gamma)$ is Zariski-dense in \mathbf{A}_G . The claim follows from Proposition 3.9. \square

The following corollary will be used repeatedly.

Corollary 3.13. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . Suppose that $\varphi_1, \varphi_2 \in \mathcal{X}(\Gamma, G)$. Then there exists $\Psi \in \text{Aut}_\mathbb{R}(\mathbf{A}_G)$ such that $\varphi_2 = \Psi \circ \varphi_1$.*

Proof. Let $\Phi_1, \Phi_2: \mathbf{A}_\Gamma \rightarrow \mathbf{A}_G$ denote the extensions of φ_1, φ_2 to the level of algebraic hulls; see Corollary 3.11. Since Φ_1 and Φ_2 are \mathbb{R} -defined isomorphisms of linear algebraic groups, and we may put $\Psi = \Phi_2 \circ \Phi_1^{-1}$. \square

3.3. Tight Lie subgroups. Throughout this section let \mathbf{A} be an \mathbb{R} -defined connected, solvable linear algebraic group. We write $\mathbf{A} = \mathbf{U} \rtimes \mathbf{T}$, where $\mathbf{U} = \text{Rad}_u(\mathbf{A})$ and \mathbf{T} is a maximal \mathbb{R} -defined torus. We remark that, if \mathbf{A} has a strong unipotent radical, then \mathbf{U} is the maximal nilpotent normal subgroup of \mathbf{A} . Let $A = \mathbf{A}_{\mathbb{R}}$ denote the associated real algebraic group, and write $A = U \rtimes T$, where $U = \mathbf{U}_{\mathbb{R}}$ is the maximal unipotent normal subgroup of A and $T = \mathbf{T}_{\mathbb{R}}$ is a maximal reductive subgroup. We observe that $[A, A] \subseteq U$.

Definition 3.14. A Lie subgroup G of A is said to be *tight* in A if it is a connected, closed subgroup of A and satisfies the following conditions: (i) G is normal in A , (ii) $\dim G = \dim U$ and (iii) G is Zariski-dense in the unipotent part of \mathbf{A} , i.e., $u(\overline{G}^z) = \mathbf{U}$, where \overline{G}^z denotes the Zariski-closure in \mathbf{A} .

We show that the set of tight Lie subgroups of A can be parametrised by Lie homomorphisms $\sigma: U \rightarrow T$ which are constant on T -orbits in U , i.e., which satisfy $[U, T] \subseteq \ker \sigma$. (A similar result appears in [10, Theorem 5.3].) As T is abelian, $[U, U] \subseteq \ker \sigma$, and hence the last condition is equivalent to $[A, A] \subseteq \ker \sigma$. For any such σ , we define

$$(3.2) \quad G_{\sigma} = \{u \sigma(u) \mid u \in U\}$$

and we observe that $\ker \sigma \subseteq G_{\sigma}$.

Proposition 3.15. *For every Lie homomorphism $\sigma: U \rightarrow T$ such that $[U, T] \subseteq \ker \sigma$ the group G_{σ} , as defined in (3.2), is a simply connected, tight Lie subgroup of A . Moreover, every tight Lie subgroup G of A is of this form.*

Proof. Let $\sigma: U \rightarrow T$ be a Lie homomorphism such that $[U, T] \subseteq \ker \sigma$, and put $G = G_{\sigma}$. Then $[A, A] \subseteq \ker \sigma \subseteq G$, and hence G is a normal subgroup of A . We observe that $U \rightarrow G$, $u \mapsto u\sigma(u)$ is a diffeomorphism. This shows that $\dim G = \dim U$ and also that G is simply connected, because being a unipotent real algebraic group, U is simply connected. It remains to check that $u(\overline{G}^z) = \mathbf{U}$. Since $[A, A] \subseteq \ker \sigma$, we have $[\mathbf{A}, \mathbf{A}] = \overline{[A, A]}^z \subseteq \overline{G}^z$. Consequently, we may assume that $\mathbf{A} = \mathbf{U} \times \mathbf{T}$. But then $u\sigma(u)$ is the Jordan decomposition for any element of G , parametrised by $u \in U$, and consequently $\mathbf{U} = \overline{U}^z \subseteq \overline{G}^z$.

Conversely, suppose that G is a tight Lie subgroup of A . For each $g \in G$, we write $g = u_g t_g$ with $u_g \in U$ and $t_g \in T$. Let $V = \{u_g \mid g \in G\}$. Since G is connected, so is its continuous image V . The fact that G is normal in A implies that V forms a subgroup of U . Being a connected Lie subgroup of a unipotent group, V is Zariski-closed in U . Consequently, we have $V = u(\overline{G}^z)_{\mathbb{R}} = U$. This shows that every $u \in U$ is of the form u_g , for some $g \in G$.

Observe that G acts on U via $u_g t_g \cdot v = u_g {}^{t_g}v$. The G -orbit of 1 in this action is equal to $V = U$ and thus $G/(G \cap T) \cong U$. Since

$\dim G = \dim U$, the map $G \rightarrow U$, $g \mapsto u_g$ is a covering map. But U is simply connected, hence it is a diffeomorphism. We can thus define a smooth map $\sigma: U \rightarrow T$ by defining $\sigma(u_g) = t_g$. Since G is a normal subgroup of A , a short computation reveals that σ is both constant on T -orbits in U and a homomorphism. \square

Corollary 3.16. *Every tight Lie subgroup of A is simply connected.*

Next we prove a relative version of Proposition 3.15.

Proposition 3.17. *Suppose that H is a tight Lie subgroup of A . Then the collection of tight Lie subgroups of A is in one-to-one correspondence with the space $\text{Hom}_{\text{Lie}}(H/[H, H], T)$ of Lie homomorphisms. Moreover, if G corresponds to $\overline{\sigma_{G|H}}$, where $\sigma_{G|H} \in \text{Hom}_{\text{Lie}}(H, T)$ with $[H, H] \subseteq \ker \sigma_{G|H}$, then $G \cap H = \ker \sigma_{G|H}$.*

Proof. According to Proposition 3.15, every tight Lie subgroup G of A is simply connected and gives rise to a Lie homomorphism $\sigma_G: U \rightarrow T$ with $[A, A] \subseteq \ker \sigma_G$. In particular, this is the case for H .

The required bijection is defined as follows. Let G be a tight Lie subgroup of A . We define a map $\sigma_{G|H}: H \rightarrow T$ as follows. Let $h \in H$. Then $h = u\sigma_H(u)$ for a unique $u \in U$. Set $\sigma_{G|H}(h) = \sigma_H(u)^{-1}\sigma_G(u)$. A short computation, using that σ_H and σ_G are constant on T -orbits, shows that the map $\sigma_{G|H}: H \rightarrow T$ is a Lie homomorphism. Since T is abelian, we have $[H, H] \subseteq \ker \sigma_{G|H}$, and clearly $G \cap H = \ker \sigma_{G|H}$.

Conversely, starting from $\sigma \in \text{Hom}_{\text{Lie}}(H, T)$ with $[H, H] \subseteq \ker \sigma$ we define $G = \{h\sigma(h) \mid h \in H\}$. A short computation shows that G is a tight Lie subgroup of A , and that the resulting map is the desired inverse. \square

Recall that a simply connected, solvable Lie group G is called unipotently connected if its maximal nilpotent normal subgroup is connected; see Section 5.

Lemma 3.18. *Let $\sigma: U \rightarrow T$ be a Lie homomorphism such that $[U, T] \subseteq \ker \sigma$. Then $G = G_\sigma$, as defined in (3.2), has the following properties:*

- (1) $A = G \rtimes T$;
- (2) $\ker \sigma = G \cap U$;
- (3) G is Zariski-dense in \mathbf{A} if and only if the group $\sigma(U)$ is Zariski-dense in \mathbf{T} .

Assuming that $G = G_\sigma$ is Zariski-dense in \mathbf{A} and that \mathbf{A} has a strong unipotent radical, the group also satisfies:

- (4) $\text{Nil}(G) = (\ker \sigma)_\circ$;
- (5) G is unipotently connected if and only if $\ker \sigma$ is connected.

Proof. By Proposition 3.15, G is a simply connected, tight Lie subgroup of A . In particular, we have $G \trianglelefteq A$. Clearly, $G \cap T = \{1\}$, $GT = UT = A$ and hence $A = G \rtimes T$. This proves (1).

Since G is tight in A , we have $\mathbf{U} \subseteq \overline{G}^z$. Hence G is Zariski-dense in \mathbf{A} if and only if $G\mathbf{U}/\mathbf{U}$ is Zariski-dense in \mathbf{A}/\mathbf{U} . The latter is equivalent to $\sigma(U)$ being Zariski-dense in \mathbf{T} . This proves (3).

Put $K = \ker \sigma$. Clearly, $K = G \cap U$, justifying (2). Now suppose that G is Zariski-dense in \mathbf{A} and that \mathbf{A} has a strong unipotent radical. Then \mathbf{U} is the maximal nilpotent normal subgroup of \mathbf{A} and consequently $K = G \cap U$ is the maximal nilpotent normal subgroup of G . This implies $\text{Nil}(G) = K_\circ$, viz. (4), and shows that G is unipotently connected if and only if K is connected, viz. (5). \square

Lemma 3.19. *Suppose that G is a connected Lie group which is contained and Zariski-dense in A . Then $[A, A]$ is contained in $\text{Nil}(G)$. In particular, G is normal in A .*

Proof. The commutator subgroup $[G, G]$ is connected and contained in $\text{Nil}(G)$. In particular, it is unipotent, and therefore Zariski-closed in A . Since G is Zariski-dense in A , it follows that $[A, A] = [G, G] \subseteq \text{Nil}(G)$. \square

Lemma 3.20. *Let G be a simply connected, solvable Lie group, and suppose that $A = A_G$ is the real algebraic hull of G . Then $G = G_\sigma$, as defined in (3.2), for a suitable Lie homomorphism $\sigma: U \rightarrow T$ such that $[U, T] \subseteq \ker \sigma$.*

Proof. By Proposition 3.15, it suffices to show that G is tight in A . By Lemma 3.19, the group G is normal in A . By the definition of the algebraic hull, we have $\dim G = \dim U$, and because G is Zariski-dense in \mathbf{A} , we surely have $\mathbf{u}(\overline{G}^z) = \mathbf{U}$. \square

4. THE FITTING SUBGROUP AND MOSTOW'S THEOREM

Let Γ be a lattice in a simply connected, solvable Lie group G . Consider the induced inclusion of real algebraic hulls $A_\Gamma \subseteq A_G$. Let U be the unipotent radical of A_Γ , which by Proposition 3.7 is also the unipotent radical of A_G . Then the closure $F = \overline{\text{Fitt}(\Gamma)}^z$ of the Fitting subgroup of Γ in A_Γ is a nilpotent normal subgroup of A_Γ . Since A_Γ has a strong unipotent radical, we have $F \leq U$. In particular, $\text{Fitt}(\Gamma)$ is a subgroup of $G \cap U = \mathbf{u}(G)$. Therefore we have

$$\text{Fitt}(\Gamma) = \mathbf{u}(\Gamma).$$

Let $N = \text{Nil}(G)$ denote the nilradical of G . Note that $N = \mathbf{u}(G)_\circ$ is the identity component of $\mathbf{u}(G)$. Mostow's theorem [18, §5], asserts that $\Gamma \cap N$ is a lattice in N . Since $\Gamma \cap N$ is a nilpotent normal subgroup of Γ , it is contained in $\text{Fitt}(\Gamma)$. In the following we strengthen Mostow's theorem: Proposition 4.3 below shows that $\text{Fitt}(\Gamma)$ is a lattice in the closed subgroup $\mathbf{u}(G)$.

We fix some further notation. Let T be a maximal reductive subgroup of A_G . Then $A_G = U \rtimes T$ is an \mathbb{R} -defined splitting of real

algebraic groups. Let $\pi: A_G \rightarrow T$ be the projection homomorphism associated to this splitting. The real algebraic torus T admits a direct decomposition $T = T_s \times T_c$, where T_s is a subtorus split over the reals and T_c is the maximal compact subgroup of T . In particular, T_s is a simply connected group. Based on the decomposition of T , we write $\pi = \pi_s \times \pi_c$ with homomorphisms $\pi_s: A_G \rightarrow T_s$ and $\pi_c: A_G \rightarrow T_c$.

Lemma 4.1. *With the notation introduced above, the following hold:*

- (1) $\pi(\Gamma)$ is discrete in T , and $\pi_s(\Gamma)$ is discrete in T_s ,
- (2) $\pi(G)$ is a closed subgroup of T ,
- (3) $\pi(\Gamma)$ is a lattice in $\pi(G)$, and $\pi_s(\Gamma)$ is a lattice in $\pi_s(G)$.

Proof. The \mathbb{Q} -defined algebraic group \mathbf{A}_Γ admits a semidirect decomposition $\mathbf{A}_\Gamma = \mathbf{U} \rtimes \mathbf{S}$ over \mathbb{Q} , where $\mathbf{U} = \text{Rad}_u(\mathbf{A}_\Gamma)$ denotes the unipotent radical and \mathbf{S} is a \mathbb{Q} -defined maximal algebraic torus. The projection $\varrho: \mathbf{A}_\Gamma \rightarrow \mathbf{S}$ associated to this decomposition is a \mathbb{Q} -defined homomorphism of algebraic groups. Fix a \mathbb{Q} -defined embedding of \mathbf{S} into a general linear group GL_n . Then the arithmetic subgroup $\mathbf{S}(\mathbb{Z}) = \mathbf{S} \cap \text{GL}_n(\mathbb{Z})$ is a discrete subgroup of $S = \mathbf{S}_\mathbb{R}$. By the additional property (iv) of the algebraic hull \mathbf{A}_Γ , the group $\Gamma \cap \varrho^{-1}(\mathbf{S}(\mathbb{Z}))$ is of finite index in Γ . Consequently, $\varrho(\Gamma)$ is discrete in S . Since $\mathbf{A}_\Gamma \leq \mathbf{A}_G$, the real algebraic torus S identifies with a subtorus of T and $\varrho(\Gamma)$ with $\pi(\Gamma)$. Therefore, $\pi(\Gamma)$ is discrete in T . Furthermore, since T_c is compact, $\pi_s(\Gamma)$, i.e., the image of $\pi(\Gamma)$ in T_s , is discrete in T_s . Thus (1) holds.

Being the continuous image of the compact space G/Γ in the Hausdorff space $T/\pi(\Gamma)$, the group $\pi(G)/\pi(\Gamma)$ is closed in $T/\pi(\Gamma)$. Hence its pre-image $\pi(G)$ under the map $T \rightarrow T/\pi(\Gamma)$ is closed in T , and (2) holds.

Since $\pi(G)/\pi(\Gamma)$ is the continuous image of the compact space G/Γ , the group $\pi(\Gamma)$ is cocompact in $\pi(G)$. Similarly, $\pi_s(\Gamma)$ is cocompact in $\pi_s(G)$. In view of (1), we deduce that (3) holds. \square

We continue to use the notation set up before Lemma 4.1. By Lemma 3.20, there exists a Lie homomorphism $\sigma: U \rightarrow T$ such that every $g \in G$ can be expressed as $g = u\sigma(u)$, in accordance with the decomposition $A_G = U \rtimes T$. We observe that $\pi(g) = \sigma(u)$ for all $g \in G$. Similarly as for π , we may decompose $\sigma = \sigma_s \times \sigma_c$ into Lie homomorphisms $\sigma_s: U \rightarrow T_s$ and $\sigma_c: U \rightarrow T_c$.

Lemma 4.2. *With the notation introduced above, we have*

$$\overline{u(G)}^z = \ker \sigma_s,$$

where $\overline{u(G)}^z$ denotes the Zariski-closure of $u(G)$ in A_G .

Furthermore, $\overline{u(G)}^z$ is a simply connected, nilpotent Lie group.

Proof. Clearly, $u(G) = U \cap G = \ker \sigma$. Since $\sigma_s: U \rightarrow T_s$ is a Lie homomorphism of simply connected groups, $\ker \sigma_s$ is a connected subgroup

of U , thus it is simply connected and Zariski-closed. Observe further that

$$\ker \sigma_s / \mathfrak{u}(G) \cong \sigma_c(\ker \sigma_s) = \pi(G) \cap T_c.$$

By (2) of Lemma 4.1, this implies that $\ker \sigma_s / \mathfrak{u}(G)$ is compact. Being a cocompact subgroup of the unipotent real algebraic group $\ker \sigma_s$, the group $\mathfrak{u}(G)$ is Zariski-dense in $\ker \sigma_s$. \square

Proposition 4.3. *Let Γ be a lattice in a simply connected, solvable Lie group G . Then, taking Zariski-closures in A_Γ , the following hold:*

- (1) $\overline{\text{Fitt}(\Gamma)} = \Gamma \cap \overline{\mathfrak{u}(G)}^z$ is a lattice in $\overline{\mathfrak{u}(G)}^z$,
- (2) $\overline{\text{Fitt}(\Gamma)}^z = \overline{\mathfrak{u}(G)}^z$,
- (3) $\text{Fitt}(\Gamma)$ is a lattice in $\mathfrak{u}(G)$.

Proof. By (3) of Lemma 4.1, the group $\Gamma(\ker \pi_s \cap G)$ is closed in G . Hence, by [22, Theorem 1.13], the group $\ker \pi_s \cap \Gamma$ is a lattice in the simply connected group $\ker \pi_s \cap G$. For $g = u\sigma(u) \in G$ we have $g \in \ker \pi_s$ if and only if $u \in \ker \sigma_s$. Thus we deduce from Lemma 3.4 and Lemma 4.2 that

$$\text{rk}(\ker \pi_s \cap \Gamma) = \dim(\ker \pi_s \cap G) = \dim(\ker(\sigma_s)) = \dim(\overline{\mathfrak{u}(G)}^z).$$

Observe further that $\pi(\ker \pi_s \cap \Gamma) = \pi(\Gamma) \cap T_c$ is finite, since $\pi(\Gamma)$ is discrete in T , by Lemma 4.1. Therefore, $\text{Fitt}(\Gamma) = \mathfrak{u}(\Gamma) = \ker \pi \cap \Gamma$ is of finite index in $\ker \pi_s \cap \Gamma$. This implies that

$$\text{rk Fitt}(\Gamma) = \text{rk}(\ker \pi_s \cap \Gamma) = \dim(\overline{\mathfrak{u}(G)}^z).$$

As $\text{Fitt}(\Gamma) = \mathfrak{u}(\Gamma) \subseteq \overline{\mathfrak{u}(G)}^z$, the Fitting subgroup $\text{Fitt}(\Gamma)$ is a discrete subgroup of the simply connected, nilpotent group $\overline{\mathfrak{u}(G)}^z$; see Lemma 4.2. Hence Lemma 3.4 shows that $\text{Fitt}(\Gamma)$ is a lattice in $\overline{\mathfrak{u}(G)}^z$, giving (1) and (2). The proof of Lemma 4.2 showed that $\mathfrak{u}(G)$ is cocompact in $\overline{\mathfrak{u}(G)}^z$, implying (3). \square

5. UNIPOTENTLY CONNECTED GROUPS

Let G be a simply connected, solvable Lie group with algebraic hull \mathbf{A}_G . As \mathbf{A}_G is solvable, we have $\text{Rad}_u(\mathbf{A}_G) = \mathfrak{u}(\mathbf{A}_G)$.

Definition 5.1. The Lie group G is called *unipotently connected*, if the closed subgroup $\mathfrak{u}(G) = G \cap \text{Rad}_u(\mathbf{A}_G)$ of G is connected.

Since \mathbf{A}_G has a strong unipotent radical, $\mathfrak{u}(G)$ is in fact the maximal nilpotent normal subgroup (the discrete nilradical) of G . Therefore, equivalently, G is unipotently connected if and only if $\mathfrak{u}(G) = \text{Nil}(G)$. This shows that the class of unipotently connected groups coincides with the class of groups (A) introduced by Starkov; see [26, Proposition 1.12]. Groups of type (A) are defined in terms the eigenvalues of the adjoint representation.

The following lemma shows that the group theoretic structure of lattices in unipotently connected groups is slightly restricted.

Lemma 5.2. *Let G be a simply connected, solvable Lie group which is unipotently connected. Then for every lattice Γ in G the Fitting quotient $\Gamma/\text{Fitt}(\Gamma)$ is torsion-free.*

Proof. In fact, $\text{Fitt}(\Gamma) = \Gamma \cap \mathfrak{u}(G)$, see the beginning of Section 4. Since G is unipotently connected, $\text{Fitt}(\Gamma) = \Gamma \cap \text{Nil}(G)$, and therefore $\Gamma/\text{Fitt}(\Gamma)$ embeds into the abelian vector group $V = G/\text{Nil}(G)$. \square

This provides a genuine restriction. For instance, using the notation of Example 2.1, the lattice $\Gamma = (\mathbb{Z} + \mathbb{Z}i).X(\mathbb{Z}[\frac{1}{2}])$ of the simply connected group $G = V.X(\mathbb{R}) \cong \widetilde{E(2)^+}$ has Fitting quotient $\Gamma/\text{Fitt}(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$. We also remark that as a side product of the construction given in Example 2.5 we obtain the following. In the notation used there, the Zariski-dense lattice $\Delta = \eta(\Lambda)X(\mathbb{Z})Y(2\mathbb{Z})$ of the Lie group G is not unipotently connected, despite $\Delta/\text{Fitt}(\Delta)$ being torsion-free. Such examples can also be constructed in an easier fashion, starting from the example of Auslander used in Example 2.3.

Proposition 5.3. *Let Γ be a lattice in a simply connected, solvable Lie group G . Then Γ has a finite index subgroup which embeds as a Zariski-dense lattice into a simply connected, solvable Lie group H which, in addition, is unipotently connected.*

Proof. In fact, Γ has a finite index subgroup Δ such that $\Delta/\text{Fitt}(\Delta)$ is torsion-free and $\Delta \leq (A_\Delta)_\circ$ is contained in the identity component of its real algebraic hull. The claim follows by a simple construction which is exhibited in [1, Chap. III §7, p. 250–251]. See also [10, Proposition 4.1], or [3, Chap. 1, Proposition 1.15] for special cases. \square

From Proposition 4.3 we derive the following corollary.

Corollary 5.4. *Let G be a simply connected, solvable Lie group, and let Γ be a lattice in G . Then the following conditions are equivalent:*

- (1) G is unipotently connected,
- (2) $\text{rk } \text{Fitt}(\Gamma) = \dim \text{Nil}(G)$,
- (3) $\overline{\text{Fitt}(\Gamma)}^z = \text{Nil}(G)$, where the Zariski-closure is taken in A_Γ ,
- (4) $\text{Fitt}(\Gamma)$ is a lattice in $\text{Nil}(G)$.

Proof. By definition, the group G is unipotently connected if and only if $\mathfrak{u}(G) = \text{Nil}(G)(= \mathfrak{u}(G)_\circ)$. Observe that $\mathfrak{u}(G)/\mathfrak{u}(G)_\circ$ is a discrete subgroup of the vector group $G/\text{Nil}(G)$, since $\mathfrak{u}(G)$ is a closed subgroup of G . Therefore $\mathfrak{u}(G)/\mathfrak{u}(G)_\circ$ is finitely generated (and abelian) of rank $\dim \overline{\mathfrak{u}(G)}^z - \dim \mathfrak{u}(G)_\circ$. By Proposition 4.3, the group $\text{Fitt}(\Gamma)$ is a lattice in $\mathfrak{u}(G)$, which implies that

$$\begin{aligned} \text{rk } \text{Fitt}(\Gamma) &= \dim \mathfrak{u}(G)_\circ + \text{rk}(\mathfrak{u}(G)/\mathfrak{u}(G)_\circ) \\ &= \dim \text{Nil}(G) + \text{rk}(\mathfrak{u}(G)/\mathfrak{u}(G)_\circ). \end{aligned}$$

Since $\text{rk}(\mathfrak{u}(G)/\mathfrak{u}(G)_\circ) = 0$ if and only if $\mathfrak{u}(G) = \mathfrak{u}(G)_\circ$, this shows that (1) and (2) are equivalent.

Writing $F = \overline{\text{Fitt}(\Gamma)}^z$, part (2) of Proposition 4.3 states that $F = \overline{\mathfrak{u}(G)}^z$. Therefore, if G is unipotently connected, $F = \mathfrak{u}(G) = \text{Nil}(G)$. Hence, (1) implies (3). Proposition 4.3 shows that (3) implies (4). Finally, Lemma 3.4 yields that (4) implies (2). \square

6. DESCRIPTION OF THE SPACE $\mathcal{G}(\Gamma)$

Throughout this section, let Γ be a torsion-free polycyclic group and suppose that its algebraic hull $\mathbf{A} = \mathbf{A}_\Gamma$ is connected. Write $\mathbf{A} = \mathbf{U} \rtimes \mathbf{T}$, where $\mathbf{U} = \text{Rad}_u(\mathbf{A})$ and \mathbf{T} is a maximal \mathbb{R} -defined torus. Put $A = \mathbf{A}_\mathbb{R}$, $U = \mathbf{U}_\mathbb{R}$ and $T = \mathbf{T}_\mathbb{R}$. Then $A = U \rtimes T$.

Recalling from Section 3.3, in particular Definition 3.14, the notion of a tight subgroup, we investigate the set

$$(6.1) \quad \mathcal{G}(\Gamma) = \{G \mid G \text{ a tight Lie subgroup of } A \text{ with } \Gamma \subseteq G\}.$$

Our interest in $\mathcal{G}(\Gamma)$ is founded on Proposition 8.11, which will show that $\mathcal{G}(\Gamma)$ captures all possible embeddings of Γ as a Zariski-dense lattice into simply connected, solvable Lie groups. The following discussion, in particular (6.2), already provides an indication of the relevance of $\mathcal{G}(\Gamma)$.

We remark that any connected Lie subgroup G of A containing Γ is Zariski-dense in A , because Γ is Zariski-dense in A , and it is normal in A by Lemma 3.19. Since $\dim U = \text{rk } \Gamma$, this shows that

$$\mathcal{G}(\Gamma) = \{G \mid G \text{ a connected, closed Lie subgroup of } A \\ \text{such that } \Gamma \subseteq G \text{ and } \dim G = \text{rk } \Gamma\}.$$

Throughout this section we suppose that $\mathcal{G}(\Gamma)$ is not empty.

Lemma 6.1. *Let $G \in \mathcal{G}(\Gamma)$. Then \mathbf{A} is an algebraic hull of G , and Γ is a Zariski-dense lattice in G .*

Proof. Clearly, Γ is a discrete subgroup of G . From Lemma 3.4 and $\text{rk } \Gamma = \dim G$ we deduce that Γ is a lattice in G . Since Γ is Zariski-dense in \mathbf{A} , so is G . Hence $\mathbf{A} = \mathbf{A}_G$ and Γ is Zariski-dense in G by Proposition 3.9. \square

Lemma 6.1 and Corollary 3.16 show that

$$(6.2) \quad \mathcal{G}(\Gamma) = \{G \mid G \text{ a simply connected, solvable Lie subgroup} \\ \text{of } A \text{ such that } \Gamma \text{ is a Zariski-dense lattice in } G\}.$$

Proposition 6.2. *Let $H \in \mathcal{G}(\Gamma)$. Then there is a bijection*

$$\mathcal{G}(\Gamma) \rightarrow \text{Hom}_{\text{Lie}}(H/\Gamma[H, H], T).$$

In particular, $\mathcal{G}(\Gamma)$ is either equal to $\{H\}$ or it is countably infinite.

Proof. Clearly, we may assume that H is not trivial. Proposition 3.17 implies that there is a bijection from $\mathcal{G}(\Gamma)$ onto $\text{Hom}_{\text{Lie}}(H/\Gamma[H, H], T)$, because $\Gamma \subseteq G \cap H \subseteq \ker \sigma_{G|H}$ for every $G \in \mathcal{G}(\Gamma)$. Now $H/\Gamma[H, H]$ is a compact torus. In fact, $H/\Gamma[H, H]$ is Hausdorff, since Zariski-density of Γ implies that $\Gamma[H, H]$ is closed in H ; see Lemma 3.8. Since H/Γ is compact, so is $H/\Gamma[H, H]$. Moreover, the latter quotient has dimension d , where $d = \dim H - \dim[H, H] > 0$. Let T_c be the maximal compact subtorus of T , and $t = \dim T_c$. Then, since the group $\text{Hom}_{\text{Lie}}(H/\Gamma[H, H], T)$ is isomorphic to $\text{Hom}(\mathbb{Z}^d, \mathbb{Z}^t)$ it is either a point ($t = 0$) or countably infinite. \square

Corollary 6.3. *Let $H \in \mathcal{G}(\Gamma)$. Then the Lie group H is of real type if and only if $\mathcal{G}(\Gamma) = \{H\}$.*

Proof. According to Proposition 6.2, we have $\mathcal{G}(\Gamma) = \{H\}$ if and only if the compact part T_c of the connected torus T is trivial. This is the case if and only if the algebraic group \mathbf{T} is \mathbb{R} -split. Observe that H is of real type if and only if there is a maximal torus in the Zariski-closure of its adjoint image which is \mathbb{R} -split. Since this Zariski-closure coincides with the adjoint image of \mathbf{A} (compare also the proof of Proposition 3.9), this is the case if and only if \mathbf{T} is \mathbb{R} -split. \square

We fix some further notation. Let $\mathbf{F} = \overline{\text{Fitt}(\Gamma)}^z$ denote the Zariski-closure of $\text{Fitt}(\Gamma)$ in $\mathbf{A} = \mathbf{A}_\Gamma$, and let $F = \mathbf{F}_\mathbb{R} = A \cap \mathbf{F}$. Since \mathbf{F} is a nilpotent normal subgroup of \mathbf{A} and since \mathbf{A} has a strong unipotent radical, we have $\mathbf{F} \subseteq \mathbf{U}$ and accordingly $F \subseteq U$.

Proposition 6.4. *Let $G, H \in \mathcal{G}(\Gamma)$, and suppose that G is unipotently connected. Then $\text{Nil}(H) \subseteq \text{Nil}(G)$.*

Proof. From Proposition 4.3 and Corollary 5.4 we deduce that $\text{Nil}(H) \subseteq \mathfrak{u}(H) \subseteq F = \text{Nil}(G)$. \square

Next we adapt the description in Proposition 6.2 to unipotently connected groups. For this purpose we consider the subset

$$\mathcal{G}^{\text{uc}}(\Gamma) = \{G \in \mathcal{G}(\Gamma) \mid G \text{ unipotently connected}\}$$

of all unipotently connected groups in $\mathcal{G}(\Gamma)$.

We assume in the following that $\mathcal{G}^{\text{uc}}(\Gamma) \neq \emptyset$. This condition puts further restrictions on Γ , but can always be met by passing to a finite index subgroup; see Proposition 5.3.

Proposition 6.5. *Let $H \in \mathcal{G}^{\text{uc}}(\Gamma)$, and set $N = \text{Nil}(H)$. Then the bijection $\mathcal{G}(\Gamma) \rightarrow \text{Hom}_{\text{Lie}}(H/\Gamma[H, H], T)$ established in Proposition 6.2 induces a bijection*

$$\mathcal{G}^{\text{uc}}(\Gamma) \rightarrow \text{Hom}_{\text{Lie}}(H/\Gamma N, T).$$

In particular, if H is not of real type then $\mathcal{G}^{\text{uc}}(\Gamma)$ is countably infinite.

Proof. The proof is similar to the proof of Proposition 6.2: one simply replaces $[H, H]$ by N . This is justified by the observation that $G \in \mathcal{G}(\Gamma)$ is unipotently connected if and only if $\text{Nil}(G) = F = \text{Nil}(H)$, by Corollary 5.4. The latter is the case if and only if $\sigma_{G|H}: H \rightarrow T$ vanishes on $F = \text{Nil}(H)$. \square

If H is of real type then, by the above, $\mathcal{G}(\Gamma) = \mathcal{G}^{\text{uc}}(\Gamma) = \{H\}$. Otherwise we have the following result.

Proposition 6.6. *Let $H \in \mathcal{G}^{\text{uc}}(\Gamma)$, and assume H is not of real type. Then the following assertions are equivalent:*

- (1) $\mathcal{G}(\Gamma) = \mathcal{G}^{\text{uc}}(\Gamma)$,
- (2) $\text{Nil}(H) = [H, H]$,
- (3) $[\Gamma, \Gamma]$ has finite index in $\text{Fitt}(\Gamma)$.

Proof. The assumption that H is not of real type implies that T_c is not trivial. Hence $\text{Hom}_{\text{Lie}}(H/\Gamma[H, H], T) = \text{Hom}_{\text{Lie}}(H/\Gamma \text{Nil}(H), T)$ if and only if $\text{Nil}(H) = [H, H]$. Thus Propositions 6.2 and 6.5 show that (1) and (2) are equivalent.

By Lemma 3.8, the group $[\Gamma, \Gamma]$ is a lattice in $[H, H]$, and, by Corollary 5.4, the group $\text{Fitt}(\Gamma)$ is a lattice in $\text{Nil}(H)$. Moreover, $[H, H]$ and $\text{Nil}(H)$ are simply connected. Since $[\Gamma, \Gamma] \subseteq \text{Fitt}(\Gamma)$ and $[H, H] \subseteq \text{Nil}(H)$, Lemma 3.4 shows that (2) is equivalent to (3). \square

We obtain an interesting class of polycyclic groups which admit Zariski-dense embeddings exclusively into simply connected, solvable Lie groups which are unipotently connected.

Corollary 6.7. *Let Δ be a torsion-free polycyclic group satisfying the following conditions:*

- (1) $\Delta \subseteq (A_\Delta)_\circ$, i.e., Δ is contained in the identity component of its real algebraic hull,
- (2) $\Delta/\text{Fitt}(\Delta)$ is torsion-free,
- (3) $[\Delta, \Delta]$ is of finite index in $\text{Fitt}(\Delta)$.

Then $\mathcal{G}(\Delta) = \mathcal{G}^{\text{uc}}(\Delta) \neq \emptyset$.

Proof. The proof of Proposition 5.3, shows that $\mathcal{G}^{\text{uc}}(\Delta) \neq \emptyset$, and the claim follows from Proposition 6.6. \square

Recall that the group of Lie automorphisms $\text{Aut}(H)$ of a simply connected Lie group H is isomorphic to the real algebraic group $\text{Aut}(\mathfrak{h})$, where \mathfrak{h} is the real Lie algebra associated to H . Let H be simply connected solvable. Then every automorphism $\varphi \in \text{Aut}(H)$ extends uniquely to an \mathbb{R} -defined automorphism $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A}_H)$ of the \mathbb{R} -defined algebraic group \mathbf{A}_H .

Define

$$\text{Aut}(H)^1 = \ker (\text{Aut}(H) \longrightarrow \text{Aut}(H/\text{Nil}(H))) .$$

Proposition 6.8. *Let $G, H \in \mathcal{G}(\Gamma)$, and suppose that G is unipotently connected. Let $\varphi \in \text{Aut}(H)^1$ and let $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A})$ be its extension. Then $\Phi(G) = G$.*

Proof. Since H and G are normal in A , their nilpotent radicals $\text{Nil}(H)$ and $\text{Nil}(G)$ are normal in A . Note that we have $\text{Nil}(H) \subseteq \text{Nil}(G) = F$ by Corollary 5.4, where F is the Zariski-closure of $\text{Fitt}(\Gamma)$ in A . Clearly, $\Phi(\text{Nil}(H)) = \text{Nil}(H)$ and, since H is Zariski-dense in A , we deduce that the induced automorphism $\Phi|_{A/\text{Nil}(H)}$ satisfies $\Phi|_{A/\text{Nil}(H)} = \text{id}_{A/\text{Nil}(H)}$. Furthermore, $\text{Nil}(H) \subseteq F$ implies that $\Phi(F) = F$ and that the induced automorphism $\Phi|_{A/F}$ satisfies $\Phi|_{A/F} = \text{id}_{A/F}$. Thus we have $\Phi(G) \subseteq GF = G$. \square

Lemma 6.9. *Let H be a simply connected solvable Lie group. Then $\text{Aut}(H)^\circ \leq \text{Aut}(H)^1$.*

Proof. The Lie algebra corresponding to $\text{Aut}(H)^\circ$ is isomorphic to the derivation algebra $\text{Der}(\mathfrak{h})$ of the Lie algebra \mathfrak{h} associated to H . It is known that the solvable Lie algebra \mathfrak{h} is mapped into its nilradical by every derivation of \mathfrak{h} ; see [12, §II.7]. Thus $\text{Aut}(H)^\circ$ acts trivially on $H/\text{Nil}(H)$. \square

Under the assumptions of Proposition 6.8 we obtain a natural homomorphism $\text{Aut}(H)^\circ \rightarrow \text{Aut}(G)^\circ$. Since G is Zariski-dense this homomorphism is injective. The following corollary records that the groups contained in $\mathcal{G}_{\text{uc}}(\Gamma)$ are very similar.

Corollary 6.10. *Let $G, H \in \mathcal{G}_{\text{uc}}(\Gamma)$. Then there exists a natural isomorphism $\text{Aut}(H)^\circ \rightarrow \text{Aut}(G)^\circ$.*

7. DESCRIPTION OF THE SPACE $\mathcal{G}_{\Gamma, G}$

We keep in place the notational conventions of Section 6. In particular, Γ is a torsion-free polycyclic group whose algebraic hull $\mathbf{A} = \mathbf{A}_\Gamma$ is connected, and $A = \mathbf{A}_{\mathbb{R}}$ denotes the group of real points. Throughout we suppose that $\mathcal{G}(\Gamma) \neq \emptyset$.

In addition, we fix $G \in \mathcal{G}(\Gamma)$ and define

$$\mathcal{G}_{\Gamma, G} = \bigcup \{ \mathcal{G}(\Delta) \mid \Delta \subseteq G \text{ a lattice with } \Delta \cong \Gamma \}.$$

This collection of Lie subgroups of A relates to the space $\mathcal{X}(\Gamma, G)$ via the map

$$(7.1) \quad \mathcal{X}(\Gamma, G) \rightarrow \mathcal{G}_{\Gamma, G}, \quad \varphi \mapsto \Phi(G),$$

where $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A})$ is the unique extension of φ ; see Corollary 3.2. We note that $\Phi(G) \in \mathcal{G}(\Delta)$, where $\Delta = \varphi(\Gamma)$.

Natural actions of $\text{Aut}(G)$ on $\mathcal{X}(\Gamma, G)$ and on $\mathcal{G}_{\Gamma, G}$ are given by

$$\vartheta \cdot \varphi = \vartheta \circ \varphi \quad \text{and} \quad \vartheta \cdot H = \Theta(H),$$

where $\vartheta \in \text{Aut}(G)$ with unique extension $\Theta \in \text{Aut}_{\mathbb{R}}(\mathbf{A})$, $\varphi \in \mathcal{X}(\Gamma, G)$ and $H \in \mathcal{G}_{\Gamma, G}$. The map (7.1) is $\text{Aut}(G)$ -equivariant with respect to these actions.

By Lemma 6.9, $\text{Aut}(G)^1 = \ker(\text{Aut}(G) \rightarrow \text{Aut}(G/\text{Nil}(G)))$ is of finite index in $\text{Aut}(G)$ and define

$$c(G) = |\text{Aut}(G) : \text{Aut}(G)^1|.$$

Proposition 7.1. *Suppose that G is unipotently connected. Then*

- (1) $\text{Aut}(G)^1$ acts trivially on the image $\tilde{\mathcal{G}}_{\Gamma, G}$ of the map (7.1).
- (2) the fibres of the induced map

$$(7.2) \quad \text{Aut}(G)^1 \backslash \mathcal{X}(\Gamma, G) \rightarrow \mathcal{G}_{\Gamma, G}, \quad [\varphi]_{\text{Aut}(G)^1} \mapsto \Phi(G).$$

are finite and bounded in size by $c(G)$.

Proof. Let $H = \Phi(G)$ be in the image of the map (7.1), where Φ is the extension of $\varphi: \Gamma \rightarrow G$. Since G is unipotently connected, so is H . Let $\Theta \in \text{Aut}_{\mathbb{R}}(\mathbf{A})$ be the extension of $\vartheta \in \text{Aut}(G)^1$. Both G and H are elements of $\mathcal{G}(\varphi(\Gamma))$, and H is unipotently connected. Thus Proposition 6.8 implies that $\vartheta \cdot H = \Theta(H) = H$, showing (1).

Observe next that composition of maps provides a natural action of $\text{Aut}(H)$ on the set

$$\{\varphi: \Gamma \hookrightarrow H \mid \varphi(\Gamma) \text{ a lattice in } H \text{ and } \Phi(G) = H\},$$

where as before $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A})$ denotes the unique extension of φ . This action is transitive: given $\varphi, \psi: \Gamma \rightarrow H$ with extensions Φ, Ψ such that $\Phi(G) = \Psi(G) = H$, the map $\Phi \circ \Psi^{-1}$ restricts to an automorphism $\vartheta \in \text{Aut}(H)$ such that $\varphi = \vartheta \circ \psi$. Furthermore, if $\varphi, \psi \in \mathcal{X}(\Gamma, G)$, then $G, H \in \mathcal{G}_{\text{uc}}(\varphi(\Gamma))$ and hence Proposition 6.8 shows that, if φ and ψ lie in the same $\text{Aut}(H)^1$ -orbit, then $[\varphi]_{\text{Aut}(G)^1} = [\psi]_{\text{Aut}(G)^1}$. Therefore each fibre of the map (7.2) has size bounded by $c(H)$. \square

The following consequence is a key ingredient in the proof of our main result Theorem A.

Corollary 7.2. *Let G be a simply connected, solvable Lie group which is unipotently connected. Let Γ be a Zariski-dense lattice in G . Then the map*

$$(7.3) \quad \text{Aut}(G) \backslash \mathcal{X}(\Gamma, G) \rightarrow \text{Aut}(G) \backslash \mathcal{G}_{\Gamma, G}, \quad [\varphi] \mapsto [\Phi(G)],$$

induced by (7.1), has finite fibres, bounded in size by $c(G)$.

Proof. As an immediate consequence of Proposition 7.1 and the fact the the map (7.1) is $\text{Aut}(G)$ -equivariant we obtain that the fibres of the map (7.3) are bounded in size by $c(G)$. \square

8. PROOFS OF THEOREM A, THEOREM E AND THEIR COROLLARIES

In this section we prove all the results from Theorem A up to Corollary H, which were stated in the introduction. The proofs of Theorem I and its Corollary J are given in Section 9.

8.1. Proof of Theorem A. Throughout, let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . We fix an algebraic hull $\mathbf{A} = \mathbf{A}_\Gamma$ of Γ such that $\Gamma \subseteq G \subseteq \mathbf{A}$ and, in this construction, \mathbf{A} is also an algebraic hull for G ; see Corollary 3.11. Thus \mathbf{A} is connected and we write $\mathbf{A} = \mathbf{U} \rtimes \mathbf{T}$, where $\mathbf{U} = \text{Rad}_u(\mathbf{A})$ and the maximal algebraic torus \mathbf{T} are defined over \mathbb{R} . Put $A = \mathbf{A}_\mathbb{R}$, $U = \mathbf{U}_\mathbb{R}$ and $T = \mathbf{T}_\mathbb{R}$. Since $\mathbf{T} = \mathbf{A}/\mathbf{U}$, there is a natural induced action of $\text{Aut}_\mathbb{R}(\mathbf{A})$ on \mathbf{T} . Let

$$\text{Aut}_\mathbb{R}(\mathbf{A})^1 = \ker (\text{Aut}_\mathbb{R}(\mathbf{A}) \longrightarrow \text{Aut}_\mathbb{R}(\mathbf{T}))$$

be the kernel of the resulting map. Put

$$c(\mathbf{A}) = |\text{Aut}_\mathbb{R}(\mathbf{A}) : \text{Aut}_\mathbb{R}(\mathbf{A})^1|$$

for the index of $\text{Aut}_\mathbb{R}(\mathbf{A})^1$. By the rigidity of algebraic tori (see [5, III.8]), the identity component $\text{Aut}_\mathbb{R}(\mathbf{A})^\circ$ is contained in $\text{Aut}_\mathbb{R}(\mathbf{A})^1$. Therefore $c(\mathbf{A})$ is finite and bounded by the number of connected components of the real algebraic group $\text{Aut}_\mathbb{R}(\mathbf{A})$.

We also consider the sets

$$\begin{aligned} \mathcal{U}_\Gamma &= \{\Delta U \mid \Delta \text{ a Zariski-dense subgroup of } A \text{ isomorphic to } \Gamma\}, \\ \mathcal{N}_{\Gamma,G} &= \{\Delta \text{Nil}(G) \mid \Delta \text{ a lattice in } G \text{ isomorphic to } \Gamma\}. \end{aligned}$$

Lemma 8.1. *The set \mathcal{U}_Γ has finite size, bounded by $c(\mathbf{A})$.*

Proof. Let Δ be a Zariski-dense subgroup of A isomorphic to Γ , and choose an isomorphism $\varphi: \Gamma \rightarrow \Delta$. Note that A is also a real algebraic hull for Δ . By the uniqueness of algebraic hulls, φ extends to an \mathbb{R} -defined algebraic automorphism $\Phi: \mathbf{A} \rightarrow \mathbf{A}$ so that $\Phi(\Gamma) = \Delta$.

Every automorphism $\Phi \in \text{Aut}_\mathbb{R}(\mathbf{A})$ induces an automorphism of the \mathbb{R} -defined torus $\mathbf{A}/\mathbf{U} \cong \mathbf{T}$. Hence, if $\Phi, \Psi \in \text{Aut}_\mathbb{R}(\mathbf{A})$ belong to the same coset of $\text{Aut}_\mathbb{R}(\mathbf{A})^1$, then $\Phi(\Gamma)U = \Psi(\Gamma)U$. Hence $|\mathcal{U}_\Gamma| \leq c(\mathbf{A})$. \square

According to Proposition 3.9 and Corollary 3.12, every lattice Δ in G with $\Delta \cong \Gamma$ is Zariski-dense in A . Since $\text{Nil}(G) \subseteq U$, we obtain a map

$$(8.1) \quad \mathcal{N}_{\Gamma,G} \rightarrow \mathcal{U}_\Gamma, \quad \Delta \text{Nil}(G) \mapsto \Delta U.$$

Lemma 8.2. *If G is unipotently connected, then the map (8.1) embeds $\mathcal{N}_{\Gamma,G}$ into \mathcal{U}_Γ . In particular, $\mathcal{N}_{\Gamma,G}$ has finite size, bounded by $c(\mathbf{A})$.*

Proof. The projection $\mathbf{A} \rightarrow \mathbf{T}$ associated to the decomposition $\mathbf{A} = \mathbf{U} \rtimes \mathbf{T}$ restricts to a homomorphism $\tau: G \rightarrow T$ with $\text{Nil}(G) \subseteq \ker \tau$. If G is unipotently connected, then $\ker \tau = \text{Nil}(G)$ by Lemma 3.18, and hence the induced homomorphism

$$G/\text{Nil}(G) \rightarrow T \cong A/U$$

is injective. Therefore, the map (8.1) is injective, and Lemma 8.1 can be applied. \square

Lemma 8.3. *If G is unipotently connected, then $c(G) \leq c(\mathbf{A})$.*

Proof. Since the homomorphism $G/\text{Nil}(G) \rightarrow T \cong A/U$ is injective and has Zariski-dense image extension of automorphisms

$$\text{Aut}(G)/\text{Aut}(G)^1 \longrightarrow \text{Aut}(A)/\text{Aut}(A)^1$$

is well defined and an injective map. \square

We reformulate these results in a slightly different way. Recall that every lattice Δ in G maps to a lattice $\Delta \text{Nil}(G)/\text{Nil}(G)$ in the vector group $V = G/\text{Nil}(G)$. Two lattices in V are called commensurable if they have a common finite index subgroup. Equivalently, they are commensurable if and only if they span the same \mathbb{Q} -vector space in V .

Corollary 8.4. *Let $V = G/\text{Nil}(G)$ and let $\eta: G \rightarrow V$ denote the natural projection. Let $\pi: \mathbf{A} \rightarrow \mathbf{T}$ denote the projection associated to the decomposition $\mathbf{A} = \mathbf{U} \rtimes \mathbf{T}$. Then the following hold.*

- (1) *Let Δ_1, Δ_2 be lattices in G which are isomorphic to Γ . Then their images $\eta(\Delta_1), \eta(\Delta_2)$ in V are commensurable if their images $\pi(\Delta_1), \pi(\Delta_2)$ in \mathbf{T} are equal.*
- (2) *The set $\mathcal{V}_{\Gamma, G} = \{\eta(\Delta) \leq V \mid \Delta \subseteq G \text{ a lattice with } \Delta \cong \Gamma\}$ falls into finitely many commensurability classes; in particular, it is countable.*
- (3) *If G is unipotently connected, then the set $\mathcal{V}_{\Gamma, G}$ is finite.*

Proof. Let Δ be a lattice in G which is isomorphic to Γ . Proposition 4.3 shows that $\text{Fitt}(\Delta)$ is cocompact in $\mathfrak{u}(G)$, hence $\eta(\text{Fitt}(\Delta))$ and $\eta(\mathfrak{u}(G))$ are commensurable subgroups of V . Therefore $\eta(\Delta)$ and $\eta(\Delta \mathfrak{u}(G))$ are commensurable. Since $\mathfrak{u}(G) = G \cap \ker \pi$, this implies (1).

By Lemma 8.1, the set $\{\pi(\Delta) \mid \Delta \subseteq G \text{ a lattice with } \Delta \cong \Gamma\}$ which naturally embeds into \mathcal{U}_{Γ} is finite. Therefore, by (1), the set $\mathcal{V}_{\Gamma, G}$ consists of finitely many commensurability classes. Hence (2) holds.

Clearly, the set $\mathcal{V}_{\Gamma, G}$ admits a one-to-one correspondence to the set $\mathcal{N}_{\Gamma, G}$. Thus (3) is a direct consequence of Lemma 8.2. \square

Next we consider the set

$$\widetilde{\mathcal{G}}_{\Gamma, G} = \{\Phi(G) \mid \Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A}) \text{ such that } \Phi(\Gamma) \subseteq G\}$$

of subgroups of A , viz. the image of the map described in (7.1). We are also interested in its subset

$$\tilde{\mathcal{G}}_{\Gamma,G}^{\text{Nil}} = \{H \in \tilde{\mathcal{G}}_{\Gamma,G} \mid \text{Nil}(H) = \text{Nil}(G)\}.$$

Proposition 8.5. *If G is unipotently connected, then $\tilde{\mathcal{G}}_{\Gamma,G}^{\text{Nil}}$ is finite, bounded in size by $c(\mathbf{A})^2$.*

Proof. Suppose that G is unipotently connected. By Corollary 5.4, the groups $\text{Fitt}(\Gamma)$ and $\text{Nil}(G)$ have the same Zariski-closure $\mathbf{F} = \overline{\text{Fitt}(\Gamma)}^{\mathbf{A}} = \overline{\text{Nil}(G)}^{\mathbf{A}}$ in \mathbf{A} , and $F = \mathbf{F}_{\mathbb{R}} = \text{Nil}(G)$. By Lemma 8.2 the set $\mathcal{N}_{\Gamma,G}$ is finite, of size bounded by $c(\mathbf{A})$. Hence it suffices to show that the set

$$(8.2) \quad \{\Phi(G) \mid \Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A}) \text{ such that } \Phi(\Gamma F) = \Gamma F \text{ and } \Phi(F) = F\}$$

is finite, of size bounded by $c(\mathbf{A})$.

The collection of all $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A})$ with $\Phi(\Gamma F) = \Gamma F$ and $\Phi(F) = F$ forms a subgroup $\text{Aut}_{\mathbb{R}}(\mathbf{A}, \Gamma F)$ of $\text{Aut}_{\mathbb{R}}(\mathbf{A})$. Let $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A}, \Gamma F) \cap \text{Aut}_{\mathbb{R}}(\mathbf{A})^1$. Then Φ induces the identity morphism on \mathbf{A}/U . We observe that

$$\Gamma U/U \cong \Gamma/(\Gamma \cap U) = \Gamma/\text{Fitt}(\Gamma) = \Gamma/(\Gamma \cap F) \cong \Gamma F/F$$

and that this chain is Φ -equivariant. Since Φ acts trivially on $\Gamma U/U \subseteq A/U$, it acts trivially on $\Gamma F/F$. Since Γ is Zariski-dense in \mathbf{A} , this shows that Φ induces the identity morphism on \mathbf{A}/\mathbf{F} . We infer that $\Phi(G) \subseteq GF = G$. This shows that the set (8.2) is finite and bounded in size by $c(\mathbf{A})$. \square

Lemma 8.6. *If G is unipotently connected, then $\tilde{\mathcal{G}}_{\Gamma,G}^{\text{Nil}} = \tilde{\mathcal{G}}_{\Gamma,G}$.*

Proof. Suppose that G is unipotently connected. We need to show that $\tilde{\mathcal{G}}_{\Gamma,G}^{\text{Nil}} \supseteq \tilde{\mathcal{G}}_{\Gamma,G}$. Let $H \in \tilde{\mathcal{G}}_{\Gamma,G}$. Then $H = \Phi(G)$ for $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A})$ such that $\Delta = \Phi(\Gamma)$ is a lattice in G and in H . Recall that \mathbf{A} is an algebraic hull of G . Thus \mathbf{A} is an algebraic hull also of Δ and of H . We observe that $\Phi(G \cap U) = H \cap U$. Since G is unipotently connected, so is H . Thus Proposition 6.4, applied to $G, H \in \mathcal{G}(\Delta)$, yields $\text{Nil}(H) = \text{Nil}(G)$. \square

From Proposition 8.5 and Lemma 8.6 we deduce the following.

Corollary 8.7. *If G is unipotently connected, then $\tilde{\mathcal{G}}_{\Gamma,G}$ is finite, and bounded in size by $c(\mathbf{A})^2$.*

Proof of Theorem A. By Corollary 7.2, the natural map (7.2) from $\text{Aut}(G)^1 \backslash \mathcal{X}(\Gamma, G)$ onto $\tilde{\mathcal{G}}_{\Gamma,G}$ has finite fibres, bounded in size by $c(G)$. By Corollary 8.7, the image of (7.2) has at most $c(\mathbf{A})^2$ elements. Hence, $\text{Aut}(G)^1 \backslash \mathcal{X}(\Gamma, G)$ and therefore also its quotient $\text{Aut}(G) \backslash \mathcal{X}(\Gamma, G)$ have at most $c(\mathbf{A})^2 c(G)$ elements. \square

Proposition 8.8. *The constants $c(G)$ and $c(\mathbf{A})$ are bounded by a constant which depends only on $\dim \text{Nil}(G)$.*

Proof. By Lemma 8.3, $c(G) \leq c(\mathbf{A})$. By definition, $c(\mathbf{A})$ is the cardinality of a finite subgroup of the algebraic automorphism group of a maximal torus \mathbf{T} in \mathbf{A} . The algebraic automorphism group of \mathbf{T} is isomorphic to the group of integral matrices $\text{GL}(n, \mathbb{Z})$, where $n = \dim T$, see [5, Chapter III.8]. The cardinality of a finite subgroup in $\text{GL}(n, \mathbb{Z})$ is bounded by a constant depending on n only, as is known classically [16]. Observe further that \mathbf{T} admits a faithful representation on the complexification of the Lie algebra of the nilradical of G . Therefore, $n = \dim \mathbf{T}$ can be bounded in terms of $\dim \text{Nil}(G)$. \square

8.2. Proofs of Corollaries B, C and D. Throughout, let Γ be a torsion-free polycyclic group with algebraic hull $\mathbf{A} = \mathbf{A}_\Gamma$. The Fitting subgroup $\text{Fitt}(\Gamma)$ is a characteristic subgroup of the polycyclic group Γ . We set

$$\text{Aut}^\circ(\Gamma) = \text{C}_{\text{Aut}(\Gamma)}(\Gamma / \text{Fitt}(\Gamma)),$$

i.e., $\text{Aut}^\circ(\Gamma)$ is the group of all automorphisms of Γ which induce the identity on the Fitting quotient $\Gamma / \text{Fitt}(\Gamma)$.

Lemma 8.9. *Let Γ be a lattice in a simply connected, solvable Lie group G . Then the group $\text{Aut}^\circ(\Gamma)$ has finite index in $\text{Aut}(\Gamma)$.*

Proof. Since every $\varphi \in \text{Aut}(\Gamma)$ extends to an automorphism $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A})$, we obtain an embedding $\text{Aut}(\Gamma) \hookrightarrow \text{Aut}_{\mathbb{R}}(\mathbf{A})$. Since $|\text{Aut}_{\mathbb{R}}(\mathbf{A}) : \text{Aut}_{\mathbb{R}}(\mathbf{A})^\circ| < \infty$, it will be enough to show that

$$\text{Aut}^\circ(\Gamma) \supseteq \{\varphi \in \text{Aut}(\Gamma) \mid \Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A})^\circ\}.$$

Let $\varphi \in \text{Aut}(\Gamma)$ with extension $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A})^\circ$. By the rigidity of tori, Φ induces the identity on \mathbf{A}/\mathbf{U} , and by Proposition 4.3 we have $\text{Fitt}(\Gamma) = \Gamma \cap \mathbf{U}$. Hence φ induces the identity on $\Gamma / \text{Fitt}(\Gamma)$.

Alternatively, the claim can be derived as follows. Let $\text{Inn}(\Gamma)$ denote the group of inner automorphisms of Γ . By [4, Theorem 1.3], the group $\text{Inn}(\Gamma) \text{Aut}^\circ(\Gamma)$ is of finite index in $\text{Aut}(\Gamma)$, for any polycyclic group Γ . Now since Γ is a lattice in G , we deduce that $[\Gamma, \Gamma] \leq \Gamma \cap [G, G] \leq \Gamma \cap \text{Nil}(G) \leq \text{Fitt}(\Gamma)$. This implies $\text{Inn}(\Gamma)$ is contained in $\text{Aut}^\circ(\Gamma)$. \square

Lemma 8.10. *Every $\varphi \in \text{Aut}^\circ(\Gamma)$ extends uniquely to a \mathbb{Q} -defined automorphism Φ of the algebraic group \mathbf{A} , and this extension satisfies $\Phi_{\mathbf{A}/\mathbf{F}} = \text{id}_{\mathbf{A}/\mathbf{F}}$, where $\mathbf{F} = \overline{\text{Fitt}(\Gamma)}^{\mathbb{Z}}$ is the Zariski-closure in \mathbf{A} .*

Proof. Let $\varphi \in \text{Aut}^\circ(\Gamma)$. Since \mathbf{A} is an algebraic hull of Γ , there is a unique extension $\Phi \in \text{Aut}(\mathbf{A})$ which is \mathbb{Q} -defined. Consider the natural projection $\mathbf{A} \rightarrow \mathbf{A}/\mathbf{F}$. Since Γ is Zariski-dense in \mathbf{A} , its image $\Gamma\mathbf{F}/\mathbf{F}$ is Zariski-dense in \mathbf{A}/\mathbf{F} . Clearly, we have $\text{Fitt}(\Gamma) = \Gamma \cap \mathbf{F}$. Since $\varphi \in \text{Aut}^\circ(\Gamma)$, this implies that $\Phi_{\mathbf{A}/\mathbf{F}}$ acts as the identity on $\Gamma\mathbf{F}/\mathbf{F}$ and hence $\Phi_{\mathbf{A}/\mathbf{F}} = \text{id}_{\mathbf{A}/\mathbf{F}}$. \square

Proof of Corollary B. Let Γ be a Zariski-dense lattice in G , where G is unipotently connected. We claim that every $\varphi \in \text{Aut}^\circ(\Gamma)$ extends to an automorphism of G .

Let $\varphi \in \text{Aut}^\circ(\Gamma)$. By Corollary 3.11 we may assume that G is contained in \mathbf{A} . By Lemma 8.10, the extension $\Phi \in \text{Aut}_{\mathbb{R}}(\mathbf{A})$ induces the identity on \mathbf{A}/\mathbf{F} , where $\mathbf{F} = \overline{\text{Fitt}(\Gamma)}^{\mathbb{Z}}$. Since G is unipotently connected, we have $\mathbf{F}_{\mathbb{R}} = \text{Nil}(G)$ according to Corollary 5.4. This implies that Φ induces the identity on $\mathbf{A}/\overline{\text{Nil}(G)}^{\mathbb{Z}}$. Therefore, we have $\Phi(G) \subseteq G \text{Nil}(G) = G$. \square

As explained in the introduction, Corollaries C and D are direct consequences of Corollary B, Proposition 5.3 and Proposition 6.6.

8.3. One-to-one correspondence between $\mathcal{S}^Z(\Gamma)$ and $\mathcal{G}(\Gamma)$. Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group. Recall from the introduction that the *structure set* $\mathcal{S}^Z(\Gamma)$ consists of equivalence classes $[\varphi]_{\mathcal{S}^Z(\Gamma)}$ of embeddings

$$\varphi: \Gamma \hookrightarrow G_\varphi$$

of Γ as a Zariski-dense lattice into simply connected, solvable Lie groups. Two embeddings $\varphi: \Gamma \hookrightarrow G_\varphi$ and $\psi: \Gamma \hookrightarrow G_\psi$ represent the same element in $\mathcal{S}^Z(\Gamma)$, if there exists an isomorphism of Lie groups $\vartheta: G_\varphi \rightarrow G_\psi$ such that $\vartheta \circ \varphi = \psi$.

By Corollary 3.11, every $\varphi: \Gamma \hookrightarrow G_\varphi$ as above admits a unique extension $\Phi: \mathbf{A}_\Gamma \rightarrow \mathbf{A}_{G_\varphi}$ to the level of algebraic hulls, yielding an isomorphism of algebraic groups. It is easy to verify that this yields a map $\varphi \mapsto \Phi^{-1}(G_\varphi)$ into $\mathcal{G}(\Gamma)$ which is constant on equivalence classes $[\varphi]_{\mathcal{S}^Z(\Gamma)}$. Thus we obtain the following *structure map* for Zariski-dense lattice embeddings of Γ :

$$\varepsilon: \mathcal{S}^Z(\Gamma) \rightarrow \mathcal{G}(\Gamma), \quad [\varphi]_{\mathcal{S}^Z(\Gamma)} \mapsto \Phi^{-1}(G_\varphi).$$

Let G be a simply connected, solvable Lie group which contains Γ as a Zariski-dense lattice. Then the deformation space $\mathcal{D}(\Gamma, G) = \text{Aut}(G) \backslash \mathcal{X}(\Gamma, G)$ admits a natural embedding

$$\mathcal{D}(\Gamma, G) \hookrightarrow \mathcal{S}^Z(\Gamma), \quad [\varphi]_{\text{Aut}(G)} \mapsto [\varphi]_{\mathcal{S}^Z(\Gamma)}$$

into the structure set $\mathcal{S}^Z(\Gamma)$. We observe that the image of the embedded $\mathcal{D}(\Gamma, G)$ under ε is the set

$$\mathcal{G}(\Gamma)_G = \{H \in \mathcal{G}(\Gamma) \mid H \cong G\};$$

under the name $S(G, \Gamma)$ the latter set plays a central role in [26].

We summarise these facts as follows.

Proposition 8.11. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . Then the structure map*

$$\varepsilon: \mathcal{S}^Z(\Gamma) \rightarrow \mathcal{G}(\Gamma)$$

is a bijection which maps the deformation space $\mathcal{D}(\Gamma, G) \subseteq \mathcal{S}^Z(\Gamma)$ onto the subset $\mathcal{G}(\Gamma)_G \subseteq \mathcal{G}(\Gamma)$.

Proof. For every $H \in \mathcal{G}(\Gamma)$ the associated inclusion map $\iota: \Gamma \hookrightarrow H$ defines an element $\delta(H) = [\iota]_{\mathcal{S}^Z(\Gamma)}$. It is straightforward to check that the resulting map $\delta: \mathcal{G}(\Gamma) \rightarrow \mathcal{S}^Z(\Gamma)$ and ε are mutually inverse to each other. One also verifies easily that δ maps $\mathcal{G}(\Gamma)_G$ to the subspace $\mathcal{D}(\Gamma, G)$ of $\mathcal{S}^Z(\Gamma)$. \square

8.4. Proofs of Theorem E and Corollaries G, F, H. Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group. By Proposition 8.11, the structure set $\mathcal{S}^Z(\Gamma)$ can be represented by the set $\mathcal{G}(\Gamma)$ of subgroups of A_Γ . Hence the results of Section 6 lead to several applications on $\mathcal{S}^Z(\Gamma)$.

Proof of Theorem E. By Proposition 8.11 the cardinality of $\mathcal{S}^Z(\Gamma)$ is the same as the cardinality of $\mathcal{G}(\Gamma)$. Therefore, Theorem E follows directly from Proposition 6.2 and Corollary 6.3. \square

Corollary F is a direct consequence of Theorem E and Proposition 8.11.

Proof of Corollary G. Suppose that Γ is a Zariski-dense lattice in G which is not strongly rigid. Then $\mathcal{S}^Z(\Gamma)$ has more than one element. Moreover, since G is unipotently connected, Proposition 6.5 shows that the subset $\mathcal{G}_{\text{uc}}(\Gamma)$ of unipotent subgroups in $\mathcal{G}(\Gamma)$ is countably infinite. Using the identification of $\mathcal{D}(H, G)$ with $\mathcal{G}(\Gamma)_H \subseteq \mathcal{G}(\Gamma)$, Theorem A shows that, for each $H \in \mathcal{G}_{\text{uc}}(\Gamma)$, the set $\mathcal{G}(\Gamma)_H$ is finite. Since the subset $\mathcal{G}_{\text{uc}}(\Gamma)$ is infinite, we conclude that there are infinitely many pairwise non-isomorphic unipotently connected groups which are contained in $\mathcal{G}_{\text{uc}}(\Gamma)$. In particular, Γ is a Zariski-dense lattice in countably infinitely many, pairwise non-isomorphic, unipotently connected groups. \square

Corollary H is a direct consequence of Theorem A.

9. THE TOPOLOGIES ON $\mathcal{S}^Z(\Gamma)$ AND $\mathcal{D}(\Gamma, G)$

The purpose of this section is to prove Theorem I and its Corollary J. Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group. We want to use the bijection $\varepsilon: \mathcal{S}^Z(\Gamma) \rightarrow \mathcal{G}(\Gamma)$, described in Proposition 8.11, to define a topology on $\mathcal{S}^Z(\Gamma)$.

For this we briefly recall the definition of two natural topologies on the collection \mathfrak{C}_X of non-empty closed subsets of a topological space X . For any subset $S \subseteq X$ we put

$$\mathcal{B}_S = \{C \in \mathfrak{C}_X \mid C \subseteq S\} \quad \text{and} \quad \mathcal{B}'_S = \{C \in \mathfrak{C}_X \mid C \cap S \neq \emptyset\}.$$

A base for the Vietoris topology on \mathfrak{C}_X is given by all finite intersections of sets taking the form \mathcal{B}_U or \mathcal{B}'_U , where U is any open subset of X .

The Chabauty topology on \mathfrak{C}_X is given by all finite intersections of sets taking the form $\mathcal{B}_{X \setminus K}$ or \mathcal{B}'_U , where K is any compact subset K and U any open subset of X . Clearly, if X is Hausdorff, then every Chabauty-open subset of \mathfrak{C}_X is also Vietoris-open. If X is compact, then the Vietoris and the Chabauty topology coincide. The Chabauty topology plays an important role in investigating the set $\mathcal{C}(G)$ of closed subgroups of a locally compact group G ; for instance, see [25] for a general discussion.

Returning attention to the lattice Γ and its structure set $\mathcal{S}^Z(\Gamma)$, let $A = A_\Gamma$ be the real algebraic hull of Γ . We recall from Section 6 that $\mathcal{G}(\Gamma)$ consists of all tight Lie subgroups of A containing Γ . Thus $\mathcal{G}(\Gamma) \subseteq \mathcal{C}(A)$ can be equipped with the Vietoris or the Chabauty subspace topology. It is natural to use the bijection $\varepsilon: \mathcal{S}^Z(\Gamma) \rightarrow \mathcal{G}(G)$ to transfer these topologies from $\mathcal{G}(G)$ to $\mathcal{S}^Z(\Gamma)$, and it turns out that both yield the discrete topology.

Proposition 9.1. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group. Then the structure set $\mathcal{S}^Z(\Gamma)$ is discrete with respect to the Chabauty topology inherited from $\mathcal{G}(\Gamma)$. The same holds for the Vietoris topology.*

Proof. Since $A = A_\Gamma$ is Hausdorff, it suffices to consider the Chabauty topology. In fact we will be working with the topology on $\mathcal{C}(A)$ which admits as a base the collection of finite intersections of sets of the form \mathcal{B}'_U , where U is an open subset of A .

Fix a point $H \in \mathcal{G}(\Gamma)$. We are to show that H is an isolated point. The group A decomposes into a semidirect product $A = H \rtimes T$, as described in Lemma 3.18. Let $\tau: A \rightarrow T$ be the associated projection with kernel H . Proposition 3.17 shows how $\hat{\tau}: \mathcal{C}(A) \rightarrow \mathcal{C}(T)$, $G \mapsto \tau(G)$ maps the set of tight Lie subgroups of A to $\mathcal{C}(T)$. Moreover, a tight Lie subgroup G satisfies $\hat{\tau}(G) = \{1\}$ if and only if $G = H$. Recall that a Lie group does not have small subgroups, i.e. there is an open neighbourhood of the identity which contains no Lie subgroups except the trivial group. Let V be such a neighbourhood in the Lie group T . Then $U = \tau^{-1}(V) \setminus \{1\}$ is an open subset of A and $\mathcal{B}'_U \cap \mathcal{G}(\Gamma) = \{H\}$. Hence H is isolated in $\mathcal{G}(\Gamma)$. \square

Let G be a simply connected, solvable Lie group which contains Γ as a Zariski-dense lattice. As described in Section 8.3 the deformation space $\mathcal{D}(\Gamma, G) = \text{Aut}(G) \backslash \mathcal{X}(\Gamma, G)$ admits a natural embedding into the structure set $\mathcal{S}^Z(\Gamma)$. The space $\mathcal{X}(\Gamma, G)$ carries the topology of pointwise convergence and induces the quotient topology on $\mathcal{D}(\Gamma, G)$. For $\mathcal{D}(\Gamma, G) \hookrightarrow \mathcal{S}^Z(\Gamma)$ to be continuous, we require that $\mathcal{D}(\Gamma, G)$ is discrete. We use a result of Wang to show that this is indeed the case.

Proposition 9.2. *Let Γ be a Zariski-dense lattice in a simply connected, solvable Lie group G . Then the deformation space $\mathcal{D}(\Gamma, G)$ is discrete with respect to the quotient topology inherited from $\mathcal{X}(\Gamma, G)$.*

Proof. Let $\varphi_0 \in \mathcal{X}(\Gamma, G)$. We have to show that the $\text{Aut}(G)$ -orbit of φ_0 in $\mathcal{X}(\Gamma, G)$ is open. Without loss of generality we may assume that $\varphi_0 = \text{id}_\Gamma: \Gamma \rightarrow G$ is the identity map on Γ . We have $\Gamma \subseteq G \subseteq A = \mathbf{A}_\mathbb{R}$, where $\mathbf{A} = \mathbf{A}_G = \mathbf{A}_\Gamma$ denotes a common algebraic hull for the groups Γ and G . Setting $N = \text{Nil}(G)$, let $D = \text{Aut}(\Gamma N)_\circ$ denote the identity component of the group of all Lie automorphisms of the group ΓN . In [27] Wang proved that the restriction map

$$D \rightarrow \mathcal{X}(\Gamma, G), \quad \vartheta \mapsto \vartheta|_\Gamma = \vartheta \circ \varphi_0$$

yields a homeomorphism between D and $\mathcal{X}(\Gamma, G)_0$, the connected component of $\mathcal{X}(\Gamma, G)$ containing φ_0 .

Every $\varphi \in \mathcal{X}(\Gamma, G)$ extends uniquely to an element $\Phi \in \text{Aut}_\mathbb{R}(\mathbf{A})$. Hence we obtain a map $D \rightarrow \text{Aut}_\mathbb{R}(\mathbf{A})$ given by $\vartheta \mapsto \Theta$, where Θ extends $\vartheta|_\Gamma$. We contend that every Θ arising in this way restricts to an automorphism of G . Let $\vartheta \in D$ and consider the extension Θ of $\vartheta|_\Gamma$. Since D is connected, ϑ acts trivially on the discrete space $\Gamma N/N$. Since Γ is Zariski-dense in \mathbf{A} , this implies that Θ acts trivially on \mathbf{A}/\mathbf{N} , where \mathbf{N} denotes the Zariski-closure of N in \mathbf{A} . Thus $\Theta(G) \subset GN = G$.

This shows that the $\text{Aut}(G)$ -orbit of φ_0 in $\mathcal{X}(\Gamma, G)$ contains the entire component $\mathcal{X}(\Gamma, G)_0$. Since $\mathcal{X}(\Gamma, G)$ is locally path connected, $\mathcal{X}(\Gamma, G)_0$ is open in $\mathcal{X}(\Gamma, G)$. Hence we conclude that the $\text{Aut}(G)$ -orbit of φ_0 in $\mathcal{X}(\Gamma, G)$ is open. \square

Theorem I is a direct consequence of Propositions 9.1 and 9.2. We also record the following conclusion from the proof of Proposition 9.2.

Corollary 9.3. *Under the same assumptions as in Proposition 9.2, the $\text{Aut}(G)$ -orbits on $\mathcal{X}(\Gamma, G)$ are unions of connected components of $\mathcal{X}(\Gamma, G)$. In particular, they are open.*

The work of Wang [27] implies that, for any lattice Γ in a simply connected, solvable Lie group the connected components of $\mathcal{X}(\Gamma, G)$ are manifolds. If a Lie group H acts transitively on a connected manifold Y then its identity component H_\circ also acts transitively on Y . Thus Corollary J follows from Corollary 9.3.

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